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The Jacobian determinant revisited

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1 Introduction

This paper is devoted to the study of the Jacobian determinant of a map g from Ω, a smooth bounded open subset of $\mathbb{R}^N$, into $\mathbb{R}^N$ ($N \geq 2$). More generally, Ω could be a smooth bounded open subset of an N-dimensional manifold. Starting with the seminal work of C.B. Morrey [29], Y. Reshetnyak [34], and J. Ball [1], it has been known that one can define the distributional Jacobian determinant $\operatorname{Det}(\nabla g)$ under fairly weak assumptions on g; in particular, it is defined for all maps $g \in W^{1,\frac{N^2}{N+1}}(\Omega)$ and also for all maps $g \in L^\infty(\Omega) \cap W^{1,N-1}(\Omega)$ (see e.g., [1, 2, 16], and [18]). Moreover

$$\left|\langle \operatorname{Det}(\nabla g), \psi\rangle\right| \leq C \min\left\{\|\nabla g\|^N_{L^{\frac{N^2}{N+1}}}, \|g\|_{L^\infty}\|\nabla g\|^{N-1}_{L^{N-1}}\right\}\|\nabla\psi\|_{L^\infty},$$
$$\forall \psi \in C^1_c(\Omega). \tag{1.1}$$

Estimate (1.1) follows from the divergence structure of the Jacobian determinant which is originally due to Morrey [29, Lemma 4.4.6]. Namely if g is smooth, we have

$$\det(\nabla g) = \sum_{j=1}^N \frac{\partial g_i}{\partial x_j} C_{i,j} = \sum_{j=1}^N \frac{\partial}{\partial x_j}[g_i C_{i,j}], \quad \forall i = 1, \ldots, N, \tag{1.2}$$

since

$$\sum_j \frac{\partial C_{i,j}}{\partial x_j} = 0, \quad \forall i = 1, 2, \ldots, N.$$

Here (∇g) is the matrix whose components are $(\nabla g)_{i,j} = \frac{\partial g_i}{\partial x_j}$, and $C = (C_{i,j})$ is the matrix of cofactors of matrix (∇g). Consequently, for smooth g, we have

$$\langle \operatorname{Det}(\nabla g), \psi\rangle = -\int_\Omega g_i F_i(g, \psi), \tag{1.3}$$

where

$$F_i(g, \psi) = \det(\nabla g_1, \ldots, \nabla g_{i-1}, \nabla\psi, \nabla g_{i+1}, \ldots, \nabla g_N),$$

for all $i = 1, \ldots, N$, and for all $\psi \in C^1_c(\Omega)$. Here we use the fact that

$$\sum_{j=1}^N \frac{\partial \psi}{\partial x_j} C_{i,j} = \det(\nabla g_1, \ldots, \nabla g_{i-1}, \nabla\psi, \nabla g_{i+1}, \ldots, \nabla g_N),$$
$$\forall i = 1, \ldots, N.$$

In particular, if g is smooth on Ω and $\psi \in C^1_c(\Omega)$, the quantity

$$\int_\Omega g_i F_i(g,\psi) \quad \text{is independent of } i. \tag{1.4}$$

Set

$$p_N = \frac{N^2}{N+1}$$

and note that if $g \in W^{1,p_N}(\Omega)$, then $g \in L^{p_N^*}(\Omega)$ by Sobolev's inequality, with $p_N^* = N^2$ since $\frac{1}{p_N^*} = \frac{1}{p_N} - \frac{1}{N} (= \frac{1}{N^2})$. Therefore $\frac{1}{p_N^*} + \frac{N-1}{p_N} = 1$, and hence by Hölder's inequality,

$$g_i \det(\nabla g_1, \dots, \nabla g_{i-1}, \nabla\psi, \nabla g_{i+1}, \dots, \nabla g_N) \in L^1(\Omega), \quad \forall i = 1, \dots, N.$$

By density, it is easy to see that (1.3) still holds for $g \in W^{1,p_N}(\Omega)$. Consequently $\operatorname{Det}(\nabla g)$ is a well-defined distribution given by (1.3) (independently of i).

It is clear from (1.3) that

(a) $\operatorname{Det}(\nabla g^{(k)})$ converges to $\operatorname{Det}(\nabla g)$ in the distributional sense if $g^{(k)}$ converges to g in $W^{1,p_N}(\Omega)$.

A more striking well-known property is the fact that

(b) $\operatorname{Det}(\nabla g^{(k)})$ converges to $\operatorname{Det}(\nabla g)$ in the distributional sense if $g^{(k)}$ converges weakly to g in $W^{1,p}(\Omega)$ for some $p > p_N$.

The standard argument goes as follows. Since $g^{(k)}$ converges weakly to g in $W^{1,p}(\Omega)$ and $p > p_N$, we deduce that $g^{(k)}$ converges to g in $L^{p_N^*}(\Omega)$ (by the compactness of the embedding $W^{1,p}(\Omega) \subset L^{p_N^*}(\Omega)$). Next we see that $\det(\nabla g_1^{(k)}, \dots, \nabla g_{i-1}^{(k)}, \nabla\psi, \nabla g_{i+1}^{(k)}, \dots, \nabla g_N^{(k)})$ is bounded in $L^{p/(N-1)}$; therefore it converges weakly to some limit in $L^{p/(N-1)}(\Omega)$. In fact this limit is precisely $\det(\nabla g_1, \dots, \nabla g_{i-1}, \nabla\psi, \nabla g_{i+1}, \dots, \nabla g_N)$ (as can be seen by induction using repeatedly formula (1.3)). A very simple alternative proof, which also gives a rate of convergence, will be presented later (see (i) of Theorem 1 applied with $p = p_N$ and $q = p_N^*$).

More generally $\operatorname{Det}(\nabla g)$ is well-defined as a distribution, via formula (1.3), if $g \in W^{1,p}(\Omega) \cap L^q(\Omega)$ with $\frac{N-1}{p} + \frac{1}{q} = 1$ and $N-1 \le p \le \infty$ (note that this formula is independent of i because the validity of (1.4) extends by density to this setting). A particular case is $p = p_N$ and $q = p_N^*$. Another interesting case is $p = N-1$ and $q = +\infty$. The same method as above gives that

(c) $\operatorname{Det}(\nabla g^{(k)}) \to \operatorname{Det}(\nabla g)$ in the distributional sense if $g^{(k)} \to g$ in $L^q(\Omega)$ and $g^{(k)} \rightharpoonup g$ weakly in $W^{1,p}(\Omega)$ with $\frac{N-1}{p} + \frac{1}{q} = 1$ and $p > N-1$ (i.e., $q < +\infty$).

In the special case where $p = p_N = \frac{N^2}{N+1}$ and $q = p_N^* = N^2$ we see that if $g^{(k)} \to g$ in $L^{p_N^*}(\Omega)$ and $g^{(k)} \rightharpoonup g$ weakly in $W^{1,p_N}(\Omega)$ then

$$\mathrm{Det}(\nabla g^{(k)}) \to \mathrm{Det}(\nabla g) \quad \text{in } \mathcal{D}'(\Omega). \tag{1.5}$$

As a consequence of (c), we have

(d) $\mathrm{Det}(\nabla g^{(k)})$ converges to $\mathrm{Det}(\nabla g)$ in the distributional sense if $p > N-1$, $g^{(k)} \rightharpoonup g$ weakly in $W^{1,p}(\Omega)$ and $\sup_k \|g^{(k)}\|_{L^q} < +\infty$ for some $q > q_0$ where q_0 is defined by $\frac{N-1}{p} + \frac{1}{q_0} = 1$.

The case $q = +\infty$ and $p = N-1$ is more delicate. Indeed, if $g^{(k)} \rightharpoonup g$ weakly in $W^{1,N-1}(\Omega)$, $F_i(g^{(k)}, \psi)$ is bounded in $L^1(\Omega)$ and converges only in the sense of measures to $F_i(g, \psi)$, not in $\sigma(L^1, L^\infty)$, and this creates a difficulty since $g \in L^\infty(\Omega)$ (g need not be continuous). Nevertheless, we will prove (see Theorem 1) that

(e) $\mathrm{Det}(\nabla g^{(k)})$ converges to $\mathrm{Det}(\nabla g)$ in the distributional sense if $g \in W^{1,N-1}(\Omega) \cap L^\infty(\Omega)$ and $(g^{(k)}) \subset W^{1,N-1}(\Omega) \cap L^\infty(\Omega)$ are such that $\sup_k \|g^{(k)}\|_{W^{1,N-1}} < +\infty$ and $\lim_{k\to 0} \|g^{(k)} - g\|_{L^\infty} = 0$.

Our first result is the following

Theorem 1 *Let $N \geq 2$, $N-1 \leq p \leq +\infty$, and $1 \leq q \leq +\infty$ be such that $\frac{N-1}{p} + \frac{1}{q} = 1$. We have, for all $f, g \in W^{1,p}(\Omega, \mathbb{R}^N) \cap L^q(\Omega, \mathbb{R}^N)$, and for all $\psi \in C_c^1(\Omega, \mathbb{R})$,*

(i) $$\left|\langle \mathrm{Det}(\nabla f), \psi\rangle - \langle \mathrm{Det}(\nabla g), \psi\rangle\right| \leq C_{N,\Omega}\|f-g\|_{L^q}(\|\nabla f\|_{L^p} + \|\nabla g\|_{L^p})^{N-1}\|\nabla\psi\|_{L^\infty}$$

and

(ii) $$\begin{aligned}\left|\langle \mathrm{Det}(\nabla f), \psi\rangle - \langle \mathrm{Det}(\nabla g), \psi\rangle\right| &\leq C_{N,\Omega}\|\nabla f - \nabla g\|_{L^p}(\|\nabla f\|_{L^p} \\ &\quad + \|\nabla g\|_{L^p})^{N-2}(\|f\|_{L^q} + \|g\|_{L^q})\|\nabla\psi\|_{L^\infty}.\end{aligned}$$

Hereafter in this paper, $C_{N,\Omega}$ denotes a positive constant depending only on N and Ω; it can change from one place to another.

Surprisingly, estimate (i) in Theorem 1 seems to have gone unnoticed until now, although it illuminates the fact that $\mathrm{Det}(\nabla g)$ is continuous under weak convergence e.g. in $W^{1,p}(\Omega)$, $p > p_N$. We also point out that the estimates in

Theorem 1 (and Theorems 2, 3 below) can be written in terms of the *Wasserstein metric*

$$\|\operatorname{Det}(\nabla f)-\operatorname{Det}(\nabla g)\|_W=\sup_{\substack{\psi\in C^1_c(\Omega)\\ \|\nabla\psi\|_{L^\infty}\le 1}}\left|\langle\operatorname{Det}(\nabla f),\psi\rangle-\langle\operatorname{Det}(\nabla g),\psi\rangle\right|.$$

In the limiting case $p=N-1$ and $q=+\infty$, if $N\geq 3$, one can replace the assumption $g\in W^{1,N-1}(\Omega)\cap L^\infty(\Omega)$ by $g\in W^{1,N-1}(\Omega)\cap BMO(\Omega)$. We need to give a "robust" meaning to the quantity $\operatorname{Det}(\nabla g)$ (since it is not true anymore that $|g||\nabla g|^{N-1}\in L^1(\Omega)$). Our argument combines the technique used in the proof of Theorem 1 with the theory of R. Coifman, P.L. Lions, Y. Meyer, and S. Semmes [15, Theorem II.1]. We postpone the precise definition of $\operatorname{Det}(\nabla g)$ and state our basic estimate.

Theorem 2 *Let $N\geq 3$. For all $f,g\in W^{1,N-1}(\Omega,\mathbb{R}^N)\cap BMO(\Omega,\mathbb{R}^N)$, and for all $\psi\in C^1_c(\Omega,\mathbb{R})$, we have*

(i) $$\left|\langle\operatorname{Det}(\nabla f),\psi\rangle-\langle\operatorname{Det}(\nabla g),\psi\rangle\right|\le C_{N,\Omega}\|f-g\|_{BMO}(\|\nabla f\|_{L^{N-1}}+\|\nabla g\|_{L^{N-1}})^{N-1}\|\nabla\psi\|_{L^\infty}$$

and

(ii) $$\begin{aligned}\left|\langle\operatorname{Det}(\nabla f),\psi\rangle-\langle\operatorname{Det}(\nabla g),\psi\rangle\right|&\le C_{N,\Omega}\|\nabla f-\nabla g\|_{L^{N-1}}(\|\nabla f\|_{L^{N-1}}\\&\quad+\|\nabla g\|_{L^{N-1}})^{N-2}(\|f\|_{BMO}+\|g\|_{BMO})\|\nabla\psi\|_{L^\infty}.\end{aligned}$$

Theorems 1 and 2 will be proved in Sects. 2.1 and 2.4.

Remark 1 In view of Theorem 2 the reader may wonder whether it is possible to improve Theorem 1 and replace $\|\nabla\psi\|_{L^\infty}$ by $\|\nabla\psi\|_{BMO}$. The answer is negative. There is no constant C such that, for all $g\in C^1_c(\Omega,\mathbb{R}^N)$, and for all $\psi\in C^1_c(\Omega,\mathbb{R})$,

$$\left|\langle\operatorname{Det}(\nabla g),\psi\rangle\right|\le C\|g\|_{L^q}\|\nabla g\|^{N-1}_{L^p}\|\nabla\psi\|_{BMO},\tag{1.6}$$

where $\frac{N-1}{p}+\frac{1}{q}=1$ and $1\le N-1\le p\le\infty$. The proof is presented in Sect. 2.2.

In Sect. 2.2 we discuss the concept of "*dipole*" which turns out to be a very effective tool in the study of distributional Jacobians concentrated on "thin" sets. The dipole construction was originally introduced by H. Brezis, J.M. Coron, and E. Lieb [9]. In Sect. 2.3 we present an example, involving dipoles, which is related to a conjecture of S. Müller [32].

Remark 2 From Theorem 1 we deduce that if $g^{(k)} \to g$ in $L^q(\Omega)$ and $(\|\nabla g^{(k)}\|_{L^p})_{k\in\mathbb{N}}$ is bounded or if $g^{(k)} \to g$ in $W^{1,p}(\Omega)$ and $(\|g^{(k)}\|_{L^q})_{k\in\mathbb{N}}$ is bounded, then $\operatorname{Det}(\nabla g^{(k)})$ converges to $\operatorname{Det}(\nabla g)$ in the sense of distributions. When $1 \le N-1 \le p \le p_N$ and $\frac{N-1}{p} + \frac{1}{q} = 1$, it may happen that $(\|\nabla g^{(k)}\|_{L^p})_{k\in\mathbb{N}}$ and $(\|g^{(k)}\|_{L^q})_{k\in\mathbb{N}}$ are bounded, $g^{(k)} \to 0$ a.e., and $\operatorname{Det}(\nabla g^{(k)})$ converges in the sense of distributions to a limit T *different* from 0, e.g. a derivative of a Dirac mass (see Sect. 2.2). Such an example was already constructed by B. Dacorogna and F. Murat [16, Proof of Theorem 1] for the special case $p = p_N$ and $q = N^2$. The construction in the general case $N-1 \le p < p_N$ is more delicate and uses dipoles.

The second part of our paper is devoted to the search of an "*optimal*" space (containing all the above cases) where one can define the Jacobian determinant (note, for example, that neither $W^{1,\frac{N^2}{N+1}}(\Omega)$ nor $W^{1,N-1}(\Omega) \cap L^\infty(\Omega)$ is a subset of the other). For this purpose it is convenient to work in the fractional Sobolev spaces $W^{s,p}(\Omega)$. We postpone again the precise definition of $\operatorname{Det}(\nabla g)$ and state our basic estimate.

Theorem 3 *Let $N \ge 2$. For all f and $g \in W^{\frac{N-1}{N},N}(\Omega, \mathbb{R}^N)$, and for all $\psi \in C^1_c(\Omega, \mathbb{R})$, we have*

$$\begin{aligned} &\big|\langle \operatorname{Det}(\nabla f), \psi\rangle - \langle \operatorname{Det}(\nabla g), \psi\rangle\big| \\ &\quad \le C_{N,\Omega} |f-g|_{W^{\frac{N-1}{N},N}} \Big(|f|^{N-1}_{W^{\frac{N-1}{N},N}} + |g|^{N-1}_{W^{\frac{N-1}{N},N}}\Big) \|\nabla\psi\|_{L^\infty}. \end{aligned} \tag{1.7}$$

We recall that for $0 < s < 1$ and $p > 1$,

$$\begin{aligned} |g|_{W^{s,p}(\Omega)} &:= \left(\int_\Omega\int_\Omega \frac{|g(x)-g(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\right)^{\frac{1}{p}}, \quad \forall g \in L^p(\Omega), \\ W^{s,p}(\Omega) &:= \big\{g \in L^p(\Omega);\, |g|_{W^{s,p}} < \infty\big\}, \end{aligned}$$

and

$$\|g\|_{W^{s,p}} := \|g\|_{L^p} + |g|_{W^{s,p}}, \quad \forall g \in W^{s,p}(\Omega).$$

As usual, the space $W^{\frac12,2}(\Omega)$ is denoted $H^{\frac12}(\Omega)$.

Remark 3 The proof of Theorem 3, presented in Sect. 3.1, relies heavily on an idea of J. Bourgain, H. Brezis, and P. Mironescu [6] (see also [7]) concerning maps in $H^{\frac12}(\Omega, \mathbb{S}^1)$ where Ω is the boundary of a domain in $\mathbb{R}^3$. This idea was subsequently exploited by T. Rivière [37], and F. Hang and F.H. Lin [22].

Remark 4 Estimate (1.7) applied with $f=0$ asserts that

$$\left|\langle \mathrm{Det}(\nabla g), \psi\rangle\right| \le C_{N,\Omega}|g|^N_{W^{\frac{N-1}{N},N}}\|\nabla\psi\|_{L^\infty},$$
$$\forall g\in W^{\frac{N-1}{N},N}(\Omega,\mathbb{R}^N), \forall\psi\in C^1_{\mathrm{c}}(\Omega,\mathbb{R}). \tag{1.8}$$

Using Hahn-Banach it is standard to deduce from (1.8) that the distribution $\mathrm{Det}(\nabla g)$ has the form

$$\mathrm{Det}(\nabla g) = \sum_{i=1}^N \frac{\partial}{\partial x_i}\mu_i \quad \text{in } \mathcal{D}'(\Omega),$$

where μ_i, $i=1,\dots,N$, are bounded Radon measures on Ω and $\|\mu_i\|_{\mathcal{M}(\Omega)} \le C_{N,\Omega}|g|^N_{W^{\frac{N-1}{N},N}}$. In our situation, we have a better information about the structure of $\mathrm{Det}(\nabla g)$, namely, there exist N *functions* $h_i\in L^1(\Omega)$, $i=1,\dots,N$, such that $\|h_i\|_{L^1}\le C_{N,\Omega}|g|^N_{W^{\frac{N-1}{N},N}}$ for $i=1,\dots,N$, and

$$\mathrm{Det}(\nabla g) = \sum_{i=1}^N \frac{\partial}{\partial x_i}h_i \quad \text{in } \mathcal{D}'(\Omega);$$

see Corollary 2 in Sect. 3.1.

Such a property is a direct consequence of the divergence structure of $\mathrm{Det}(\nabla g)$ when $g\in W^{1,p}\cap L^q$ with p and q as in Theorem 1, but it is already non-obvious in the framework of Theorem 2.

However, one cannot find such functions h_i belonging to the Hardy space $\mathcal{H}^1(\Omega)$ (with an estimate of the $\mathcal{H}^1$-norm); see Remark 1.

Remark 5 W. Sickel and A. Youssfi [39, 40] have also defined a distributional Jacobian determinant for maps in a space which resembles ours. They proved that $\mathrm{Det}(\nabla g)$ is well-defined as a distribution (in the dual of $C^1_{\mathrm{c}}(\Omega)$) if $g\in H_N^{\frac{N-1}{N}}(\Omega)$ where H^s_p denotes the usual Bessel-potential space. Recall (see e.g. [41]) that $H^s_p = F^s_{p,2}$ ($F^s_{p,q}$ is the standard Lizorkin-Triebel space) and $W^{s,p} = F^s_{p,p}$ $(= B^s_{p,p})$; in addition $F^s_{p,2}(\Omega)\subset F^s_{p,p}(\Omega)$ with strict inclusion if $p>2$ (see e.g. [41, Proposition 2 on page 47]). Therefore, if $N\ge 3$, the space $H_N^{\frac{N-1}{N}}(\Omega)$ considered by W. Sickel and A. Youssfi is *strictly smaller* than the space $W^{\frac{N-1}{N},N}(\Omega)$ we use (when $N=2$, $H_2^{\frac12}(\Omega) = W^{\frac12,2}(\Omega) = H^{\frac12}(\Omega)$). Moreover, our proof is much simpler: it relies only on the fact that $W^{\frac{N-1}{N},N}$ is the trace space of $W^{1,N}$ (and on Lemma 3 below, which is just an integration by parts), while their proof is quite sophisticated and involves paraproducts.

Remark 6 We recover with Theorem 3 all the definitions of Jacobian determinants mentioned above, except the case $N = 2$, $p = 1$, and $q = \infty$. Indeed, we have

(i) $W^{1,p}(\Omega) \subset W^{\frac{N-1}{N},N}(\Omega)$ with continuous embedding if $p \geq \frac{N^2}{N+1}$ and compact embedding if $p > \frac{N^2}{N+1}$ (see e.g., [41, Sect. 3.3.1]) (this implies (a) and (b)).

(ii) $W^{1,p}(\Omega) \cap L^q(\Omega) \subset W^{\frac{N-1}{N},N}(\Omega)$ with continuous embedding if $\frac{N-1}{p} + \frac{1}{q} = 1$ $(1 \leq q \leq \infty)$ except in the case $N = 2$, $p = 1$, and $q = +\infty$. Moreover,

$$\|g\|_{W^{\frac{N-1}{N},N}} \leq C\|g\|^{\alpha}_{W^{1,p}}\|g\|^{1-\alpha}_{L^q}, \tag{1.9}$$

with $\alpha = 1 - \frac{1}{N}$ (see e.g., [11, Corollary 3.2]). This implies (c), and (e) for $N \geq 3$.

(iii) The case where $g \in W^{1,N-1}(\Omega) \cap BMO(\Omega)$ and $N \geq 3$ can also be covered by Theorem 3 using the fact that $W^{1,N-1}(\Omega) \cap BMO(\Omega) \subset W^{\frac{N-1}{N},N}(\Omega)$ with

$$\|g\|_{W^{\frac{N-1}{N},N}} \leq C\|g\|^{\alpha}_{W^{1,N-1}}\|g\|^{1-\alpha}_{BMO}, \tag{1.10}$$

for $\alpha = 1 - \frac{1}{N}$. Inequality (1.10) is probably known to the experts but we could not find a reference in the literature; therefore we have presented a proof in the Appendix.

Our next result asserts that Theorem 3 is *optimal* in the framework of the spaces $W^{s,p}$. More precisely, the distributional Jacobian is well-defined in $W^{s,p}(\Omega)$ *if and only if* $W^{s,p}(\Omega) \subset W^{\frac{N-1}{N},N}(\Omega)$.

Theorem 4 *Let $s \in (0, 1)$ and $p \in (1, +\infty)$ be such that $W^{s,p}(\Omega) \not\subset W^{\frac{N-1}{N},N}(\Omega)$. Then there exists a sequence $(g^{(k)}) \subset C^1(\bar{\Omega}, \mathbb{R}^N)$ and a function $\psi \in C^1_{\mathrm{c}}(\Omega, \mathbb{R})$ such that*

$$\lim_{k\to\infty} \|g^{(k)}\|_{W^{s,p}} = 0 \tag{1.11}$$

and

$$\lim_{k\to\infty} \int_{\Omega} \det(\nabla g^{(k)})\psi = +\infty. \tag{1.12}$$

In order to prove Theorem 4 we consider all possible cases:

(i) $s + \frac{1}{N} > \max\{1, \frac{N}{p}\}$ and then $W^{s,p}(\Omega) \subset W^{\frac{N-1}{N},N}(\Omega)$ (by the fractional Sobolev embedding, see e.g. [41, page 196]), so that the distributional Jacobian is well-defined using Theorem 3.

(ii) $s+\frac{1}{N}<\max\{1,\frac{N}{p}\}$ and then the distributional Jacobian is meaningless because one can construct a sequence $(g^{(k)})$ satisfying (1.11) and (1.12) (see Lemma 5 in Sect. 3.2.1). When $p\geq 2$, we have $H^s_p=F^s_{p,2}\subset F^s_{p,p}=W^{s,p}$; therefore we could use the example constructed in H^s_p by W. Sickel and A. Youssfi [40, Theorem 4]. However, if $p<2$, we have $W^{s,p}=F^s_{p,p}\subset F^s_{p,2}=H^s_p$ and therefore we *cannot* directly rely on the example of [40] constructed in H^s_p. Anyway, the construction we present for the case (ii) is very elementary and valid for all $p\in(1,+\infty)$.

(iii) the borderline case $s+\frac{1}{N}=\max\{1,\frac{N}{p}\}$ is more delicate. If $p\leq N$ and $s=\frac{N}{p}-\frac{1}{N}$ one knows that $W^{s,p}(\Omega)\subset W^{\frac{N-1}{N},N}(\Omega)$ (see e.g. [41, page 196]). If $p>N$ and $s=1-\frac{1}{N}$ the distributional Jacobian is again meaningless because one can exhibit a sequence $(g^{(k)})$ satisfying (1.11) and (1.12) (see Lemma 5 in Sect. 3.2.1). The construction of $g^{(k)}$ in this case is quite subtle and involves several ingredients: a suggestion of L. Tartar used in [2, Counterexample 7.3], a device communicated to us by P. Mironescu [28], and the theory of Besov spaces [38].

Using Theorem 3 and the fact that $C^{0,\alpha}(\bar{\Omega})\subset W^{\frac{N-1}{N},N}(\Omega)$ with continuous embedding when $\alpha>\frac{N-1}{N}$, we are able to define $\mathrm{Det}(\nabla g)$ for maps $g\in C^{0,\alpha}(\bar{\Omega},\mathbb{R}^N)$ with $\alpha>\frac{N-1}{N}$ and obtain:

Corollary 1 *Let $N\geq 2$ and $\frac{N-1}{N}<\alpha<1$. We have*

(i) *If $g\in C^{0,\alpha}(\bar{\Omega},\mathbb{R}^N)$ then for some positive constant $C_{N,\Omega,\alpha}$,*

$$|\langle \mathrm{Det}(\nabla g),\psi\rangle|\leq C_{N,\Omega,\alpha}|g|^N_{C^{0,\alpha}}\|\nabla\psi\|_{L^\infty},\quad \forall\psi\in C^1_c(\Omega).$$

(ii) *If $(g^{(k)})\subset C^{0,\alpha}(\bar{\Omega},\mathbb{R}^N)$ converges to some g in $C^{0,\alpha}(\bar{\Omega},\mathbb{R}^N)$, then*

$$\lim_{k\to\infty}\langle \mathrm{Det}(\nabla g^{(k)}),\psi\rangle=\langle \mathrm{Det}(\nabla g),\psi\rangle,\quad \forall\psi\in C^1_c(\Omega).$$

Corollary 1 is optimal in the framework of Hölder spaces (see Proposition 4 in Sect. 3.2.2).

In an earlier paper [12], we studied "minimal" assumptions in order to define $\mathrm{Det}(\nabla g)$ (as a distribution) for maps g from $\mathbb{S}^N$ into itself. The condition $g\in VMO(\mathbb{S}^N,\mathbb{S}^N)\cap W^{\frac{N-1}{N},N}(\mathbb{S}^N,\mathbb{S}^N)$ played there an essential role. As mentioned above, if $g:\mathbb{R}^N\to\mathbb{R}^N$, we only need the condition $g\in W^{\frac{N-1}{N},N}(\mathbb{R}^N,\mathbb{R}^N)$ (and the stringent *VMO* assumption is totally unnecessary). We point out that prior to this work, other authors were also concerned with the definition of a distributional Jacobian for maps $g:\Omega\to\mathbb{S}^{N-1}$ (we really mean $\mathbb{S}^{N-1}$, *not* $\mathbb{S}^N$), where Ω is a domain in $\mathbb{R}^N$, or an N-dimensional manifold. Of course, in this case $\mathrm{Det}(\nabla g)$ is a distribution concentrated on the singular set of g. F. Hang and F.H. Lin [22] (see also R. Jerrard and H. Soner [27]) assumed that $g \in W^{\frac{N-1}{N},N}(\Omega, \mathbb{S}^{N-1})$; they used the same idea as in an earlier paper of J. Bourgain, H. Brezis, and P. Mironescu [6] concerning the case $N = 2$. Actually, their definition extends with no change to $\mathbb{R}^N$-valued maps. We call the attention of the reader to the result of J. Bourgain, H. Brezis, and P. Mironescu [8]. In [8], they are able to define $\mathrm{Det}(\nabla g)$ for all maps g in $W^{s,p}(\Omega, \mathbb{S}^{N-1})$ for any $s \in (0, 1)$ with $sp = N - 1$; when $0 < s < (N-1)/N$ and $sp = N - 1$, the space $W^{s,p}(\Omega, \mathbb{S}^{N-1})$ is *bigger* than $W^{\frac{N-1}{N},N}(\Omega, \mathbb{S}^{N-1})$. In this case, it is important to consider $\mathbb{S}^{N-1}$-valued maps, otherwise one would have a contradiction with Theorem 4.

Finally, we mention that the Jacobian determinant was extensively studied in the literature see e.g., [1, 2, 10, 15–18, 21, 23–26, 29–32, 35, 36], and references therein.

2 Theorems 1 and 2, and related topics

2.1 Proof of Theorem 1

It suffices to prove the results for f and g smooth. Set $g = (g_1, \dots, g_N)$ and $f = (f_1, \dots, f_N)$. Write

$$\det(\nabla f) - \det(\nabla g) = \sum_{i=1}^{N} X_i,$$

where

$$\begin{cases} X_1 = \det(\nabla(f_1 - g_1), \nabla f_2, \dots, \nabla f_N), \\ X_N = \det(\nabla g_1, \nabla g_2, \dots, \nabla g_{N-1}, \nabla(f_N - g_N)), \end{cases}$$

and for $i = 2, \dots, N-1$,

$$X_i = \det(\nabla g_1, \dots, \nabla g_{i-1}, \nabla(f_i - g_i), \nabla f_{i+1}, \dots, \nabla f_N).$$

Applying (1.3) and Hölder's inequality yields

$$\left| \int_\Omega X_i \psi \right| \le \| f_i - g_i \|_{L^q} \| \nabla f \|_{L^p}^{N-i} \| \nabla g \|_{L^p}^{i-1} \| \nabla \psi \|_{L^\infty}.$$

This implies (i). To prove (ii), it suffices to note that, by (1.3),

$$\left| \int_\Omega X_i \psi \right| \le \| \nabla f_i - \nabla g_i \|_{L^p} \big(\| \nabla f \|_{L^p} + \| \nabla g \|_{L^p} \big)^{N-2} \times \big(\| f \|_{L^q} + \| g \|_{L^q} \big) \| \nabla \psi \|_{L^\infty}.$$

□

Remark 7 In the proof of Theorem 1, we implicitly use the following identity:

$$\det(\nabla f) - \det(\nabla g) = \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial}{\partial x_j}\{(f_i - g_i)M^{(i)}_{i,j}\}, \tag{2.1}$$

where $M^{(i)}$ is the matrix of cofactors of the matrix

$$(\nabla g_1, \dots, \nabla g_{i-1}, \nabla(f_i - g_i), \nabla f_{i+1}, \dots, \nabla f_N)^T.$$

Here A^T denotes the transpose of A for any matrix A.

2.2 The dipole construction. Further discussion around Theorem 1

The concept of dipole plays an important role in this section and we recall its construction.

Fix a smooth map $\omega : \mathbb{R}^{N-1} \to \mathbb{S}^{N-1}$ such that $\omega(y) = \mathcal{N} = (0, \dots, 0, 1)$ for $|y| \geq 1$ and ω covers $\mathbb{S}^{N-1}$ exactly once (ω has a degree 1 if $\mathbb{R}^{N-1}$ is identified with $\mathbb{S}^{N-1}$ via a stereographic projection). Consider the cone Q_0 in $\mathbb{R}^N$ defined by

$$Q_0 = \left\{ x = (x', z) \in \mathbb{R}^{N-1} \times \mathbb{R};\ \frac{L|x'|}{\rho z} < 1 \text{ with } 0 < z \leq L \right\},$$

with height L and spherical base of radius $\rho \leq L$. The map $f_0 : \mathbb{R}^{N-1} \times (-\infty, L) \to \mathbb{R}^N$ is defined by

$$f_0(x', z) = \begin{cases} \omega(\frac{Lx'}{\rho z}) - \mathcal{N} & \text{if } (x', z) \in Q_0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Next we perform a symmetry Σ about the hyperplane $\{(x', L);\ x' \in \mathbb{R}^{N-1}\}$ and we obtain a map $f : \mathbb{R}^N \to \mathbb{R}^N$ with support in the region $Q = Q_0 \cup \Sigma(Q_0)$. When needed we will write $f_{L,\rho}$ instead of f.

The map f is smooth except at the points $P = (0, \dots, 0)$ and $D = (0, \dots, 0, 2L)$. Moreover (see the computation below), $f \in W^{1,p}(\mathbb{R}^N)$ for all $1 \leq p < N$ and $f \in L^\infty(\mathbb{R}^N)$; f does *not* belong to $W^{1,N}(\mathbb{R}^N)$. Such an f is a good example of a map which enters in framework of Theorem 1. Multiplying f by an appropriate factor (and keeping the same notation f) we obtain, via a standard computation,

$$\mathrm{Det}(\nabla f) = \delta_P - \delta_D. \tag{2.3}$$

More generally if ν is an integer we may glue ν copies of ω in a ball of radius $R \approx \nu^{\frac{1}{N-1}}$. After rescaling we obtain a map $\omega^\nu : \mathbb{R}^{N-1} \to \mathbb{S}^{N-1}$ such that

$$\begin{cases} \operatorname{supp} \omega^\nu \subset B_1, \text{ the unit ball of } \mathbb{R}^{N-1}, \\ \|\nabla \omega^\nu\|_{L^\infty} \lesssim \nu^{\frac{1}{N-1}}, \\ \omega^\nu \text{ covers } \mathbb{S}^{N-1} \text{ exactly } \nu \text{ times.} \end{cases} \tag{2.4}$$

Hereafter in this section, $a \lesssim b$ means $a \le Cb$ for some $C > 0$ depending only on p, q, N, and Ω, $a \gtrsim b$ means $b \lesssim a$ and $a \approx b$ means $a \lesssim b$ and $b \lesssim a$.

The corresponding f^ν (defined via (2.2)) satisfies

$$\operatorname{Det}(\nabla f^\nu) = \nu(\delta_P - \delta_D).$$

Note that

$$\operatorname{vol} Q \approx \rho^{N-1} L,$$

and thus $\forall\, q \in [1, \infty)$,

$$\|f\|_{L^q}^q \approx \rho^{N-1} L \tag{2.5}$$

while

$$\|f\|_{L^\infty} = 2. \tag{2.6}$$

On the other hand, we have in Q_0,

$$\begin{aligned} |\nabla f_0(x', z)| &\lesssim \left|\nabla \omega\left(\frac{Lx'}{\rho z}\right)\right| \left(\frac{L}{\rho z} + \frac{L|x'|}{\rho z^2}\right) \\ &\le \left|\nabla \omega\left(\frac{Lx'}{\rho z}\right)\right| \left(\frac{L}{\rho z} + \frac{1}{z}\right) \le \left|\nabla \omega\left(\frac{Lx'}{\rho z}\right)\right| \frac{2L}{\rho z}, \end{aligned}$$

since $\rho \le L$. Therefore we have, provided $p < N$,

$$\int_Q |\nabla f(x', z)|^p \, dx' \, dz \lesssim \rho^{N-1-p} L. \tag{2.7}$$

For later reference note that, by (2.4),

$$\int_Q |\nabla f^\nu|^p \lesssim \rho^{N-1-p} L \nu^{\frac{p}{N-1}}$$

and in particular

$$\int_Q |\nabla f^\nu|^{N-1} \lesssim L\nu. \tag{2.8}$$

We may also glue a sequence of disjoint dipoles $([P_i, D_i])_{i\in\mathbb{N}}$ placed on the x_N-axis, with $\rho_i = L_i = \frac{1}{2}|P_i - D_i|$. For any $p < N$ we obtain a map $f \in W^{1,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ satisfying

$$\mathrm{Det}(\nabla f) = \sum_{i=1}^{\infty} (\delta_{P_i} - \delta_{D_i}) \quad \text{in } \mathcal{D}'(\mathbb{R}^N) \tag{2.9}$$

provided $\sum_i |P_i - D_i| < +\infty$ and $\sum_i |P_i - D_i|^{N-p} < +\infty$. Note that the RHS in (2.9) is a distribution and is *not a measure* (more precisely $\mathrm{Det}(\nabla f)$ belongs to the dual of C^1_c).

We may now state a result mentioned in Remark 2.

Proposition 1 *Assume p and q satisfy*

$$1 \le N - 1 \le p \le p_N \tag{2.10}$$

and

$$\frac{N-1}{p} + \frac{1}{q} = 1. \tag{2.11}$$

Then there exists a sequence $g^{(k)}$ in $C^\infty_c(\mathbb{R}^N)$ such that

$$\begin{cases} \operatorname{supp} g^{(k)} \subset B(0, r_k) \quad \textit{with } r_k \to 0 \textit{ as } k \to \infty, \\ \|\nabla g^{(k)}\|_{L^p} \le C, \\ \|g^{(k)}\|_{L^q} \le C, \\ \det(\nabla g^{(k)}) \to \frac{\partial}{\partial x_N}\delta_0 \quad \textit{in } \mathcal{D}'(\mathbb{R}^N) \textit{ as } k \to \infty. \end{cases}$$

Proof We distinguish two cases:

Case 1: $N - 1 < p \le p_N$ and $N^2 \le q < \infty$.

Case 2: $p = N - 1$ and $q = \infty$.

Case 1: $N - 1 < p \le p_N < N$ and $N^2 \le q < \infty$. Set

$$g_{L,\rho} = \frac{1}{(2L)^{\frac{1}{N}}} f_{L,\rho},$$

so that, by (2.3), we have

$$\mathrm{Det}(\nabla g_{L,\rho}) = \frac{1}{2L}(\delta_P - \delta_D). \tag{2.12}$$

From (2.5) and (2.7) we obtain

$$\|\nabla g_{L,\rho}\|^p_{L^p} \lesssim \rho^{N-1-p} L^{1-\frac{p}{N}}$$

and

$$\|g_{L,\rho}\|_{L^q}^q \lesssim \rho^{N-1} L^{1-\frac{q}{N}}.$$

It follows from (2.10) and (2.11) that

$$\gamma = \frac{1-p/N}{p-(N-1)} = \frac{q/N-1}{N-1} \geq 1.$$

Finally we choose

$$\rho = L^{\gamma}$$

and then

$$\begin{cases} \|\nabla g_{L,\rho}\|_{L^p} \lesssim 1, \qquad \|g_{L,\rho}\|_{L^q} \lesssim 1, \\ \rho \leq L \text{ provided } L \leq 1, \\ \operatorname{supp} g_{L,\rho} \subset B(0, L). \end{cases}$$

Moreover, by (2.12),

$$\operatorname{Det}(\nabla g_{L,L^{\gamma}}) \to -\frac{\partial}{\partial x_N}\delta_0 \quad \text{as } L \to 0.$$

The desired result is obtained after changing x_N into $-x_N$ and smoothing $g_{L,L^{\gamma}}$ by convolution with a sequence of mollifiers.

Case 2: $p = N-1$ and $q = \infty$. Here we set

$$g_{L,\nu} = f_{L,L}^{\nu},$$

so that

$$\operatorname{Det}(\nabla g_{L,\nu}) = \nu(\delta_P - \delta_D).$$

From (2.6) and (2.8), we have

$$\|\nabla g_{L,\nu}\|_{L^{N-1}}^{N-1} \lesssim L\nu$$

and

$$\|g_{L,\nu}\|_{L^{\infty}} \leq 2.$$

Finally we choose $L = \frac{1}{2\nu}$ and we see that

$$\operatorname{Det}(\nabla g_{\frac{1}{2\nu},\nu}) \to -\frac{\partial}{\partial x_N}\delta_0 \quad \text{as } \nu \to \infty.$$

We conclude as above. □

In view of Proposition 1 one may wonder whether there exists some g, satisfying the assumptions of Theorem 1, such that

$$\mathrm{Det}(\nabla g) = \frac{\partial}{\partial x_N}\delta_a \quad \text{with } a \in \Omega. \tag{2.13}$$

The answer is negative. Here is the reason. Without loss of generality, we may assume that $0 \in [-1, 1]^N \subset \Omega$ and a is the origin. Consider $\psi_\varepsilon(x_1, \ldots, x_N) = \psi_{1,\varepsilon}(x_N) \prod_{i=1}^{N-1} \psi_2(x_i)$ where

$$\psi_{1,\varepsilon}(x_N) = \varepsilon\psi_1(x_N/\varepsilon),$$

ψ_1, ψ_2 are smooth, $0 \le \psi_1 \le 1$, $\psi_1'(0) = -1$, $\operatorname{supp}\psi_1 \subset (-1, 1)$, $0 \le \psi_2 \le 1$, $\operatorname{supp}\psi_2 \subset (-1, 1)$, and $\psi_2 = 1$ in $(-1/2, 1/2)$. Then, by (2.13),

$$\langle \mathrm{Det}(\nabla g), \psi_\varepsilon\rangle = 1, \quad \forall 0 < \varepsilon < 1.$$

Using (1.3), we write

$$\mathrm{Det}(\nabla g) = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} h_i \quad \text{in } \mathcal{D}'(\Omega) \text{ with } h_i \in L^1(\Omega) \text{ for } i = 1, \ldots, N \tag{2.14}$$

and we deduce that

$$\left|\langle \mathrm{Det}(\nabla g), \psi_\varepsilon\rangle\right| \lesssim \varepsilon \sum_{i=1}^{N-1} \int_\Omega |h_i| + \int_{|x_N|<\varepsilon} |h_N| \to 0 \quad \text{as } \varepsilon \to 0.$$

Impossible. □

In fact one can prove the following sharper result (see also Remark 12).

Proposition 2 *Assume that* $g \in W^{1,p}(\Omega, \mathbb{R}^N) \cap L^q(\Omega, \mathbb{R}^N)$ *where* $1 \le N - 1 \le p \le +\infty$ *and* $1 \le q \le +\infty$ *satisfy*

$$\frac{N-1}{p} + \frac{1}{q} = 1.$$

Suppose that

$$\mathrm{Det}(\nabla g) = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\mu_i, \quad \textit{where } \mu_i \in \mathcal{M}(\Omega)\ \forall i = 1, \ldots, N. \tag{2.15}$$

Then, for all $i = 1, \ldots, N$, *the measures* $|\mu_i|$ *do not charge sets of zero* $W^{1,N}$*-capacity.*

In particular, there does not exist g as in Proposition 2 such that (2.13) holds (since points have zero $W^{1,N}$-capacity).

Proof Combining (2.15) and (2.14) we have

$$\operatorname{div}(\mu - h) = 0, \quad \text{where } \mu = (\mu_i) \text{ and } h = (h_i).$$

It follows from a result of J. Bourgain and H. Brezis [4, 5] (see also [42]) that

$$\nu = \mu - h \in \left(W_0^{1,N}\right)^\star = W^{-1,\frac{N}{N-1}}.$$

Therefore $|\nu|$ does not charge sets of zero $W^{1,N}$-capacity; see [20] (and also [3]). Hence $|\mu|$ has the same property since $|\mu| \le |\nu| + |h|$. □

Remark 8 Of course, the decomposition (2.15) is not unique. Under the assumptions of Theorem 1 one can write (2.15) with measures μ_i that are *not* L^1 functions. Take for example g to be a dipole, i.e., $g \in W^{1,p} \cap L^\infty$ for all $p < N$ and

$$\operatorname{Det}(\nabla g) = \delta_P - \delta_D.$$

We may also write

$$\delta_P - \delta_D = \frac{\partial}{\partial x_1}\mu_1,$$

where μ_1 is the 1-dimensional Hausdorff measure on the interval $[P, D]$.

Finally we present the

Proof of Remark 1 Suppose for simplicity that $B_1 = B(0,1) \subset \Omega$. Fix $g \in C_c^\infty(B_1)$ such that

$$\int_{B_1} \det(\nabla g(x)) x_1 \, dx = 1. \tag{2.16}$$

For $0 < \varepsilon < 1$, define

$$g_\varepsilon(x) = g(x/\varepsilon), \quad \forall x \in B_1.$$

Then

$$\int_{B_1} \det(\nabla g_\varepsilon(x))\psi(x)\, dx = \int_{B_1} \det(\nabla g(x))\psi(\varepsilon x)\, dx, \quad \forall \psi \in C_c^1(B_1), \tag{2.17}$$

and

$$\|\nabla g_\varepsilon\|_{L^p(B_1)} = \varepsilon^{(N/p)-1}\|\nabla g\|_{L^p(B_1)} \quad \text{and} \quad \|g_\varepsilon\|_{L^q(B_1)} = \varepsilon^{N/q}\|g\|_{L^q(B_1)}. \tag{2.18}$$

Since

$$\int_{B_1} \det(\nabla g) = 0 \tag{2.19}$$

by the divergence structure of the Jacobian determinant, it follows from (2.17) that

$$\int_{B_1} \det(\nabla g_\varepsilon)\psi = \varepsilon \nabla\psi(0) \cdot \int_{B_1} \det(\nabla g(x)) x\, dx + O(\varepsilon^2). \tag{2.20}$$

Assuming that (1.6) holds we have

$$\begin{aligned}\left|\int_{B_1} \det(\nabla g_\varepsilon)\psi\right| &\le C\|g_\varepsilon\|_{L^q}\|\nabla g_\varepsilon\|_{L^p}^{N-1}\|\nabla\psi\|_{BMO}\\ &\le C\varepsilon\|\nabla\psi\|_{BMO} \quad \text{by (2.18).}\end{aligned}$$

Combining with (2.20), and passing to the limit as $\varepsilon \to 0$, we obtain

$$\left|\nabla\psi(0) \cdot \int_{B_1} \det(\nabla g(x)) x\, dx\right| \le C\|\nabla\psi\|_{BMO},$$

and we deduce from (2.16) that

$$\left|\frac{\partial\psi}{\partial x_1}(0)\right| \le C\sum_{i=2}^{N}\left|\frac{\partial\psi}{\partial x_i}(0)\right| + C\|\nabla\psi\|_{BMO}. \tag{2.21}$$

Choosing in (2.21) the function $\psi(x) = x_1\eta(x)\ln(|x|^2 + \delta^2)^{-1}$, where $\eta \in C_c^1(B_1)$ and $\eta(x) = 1$ near 0, yields $|\ln\delta| \le C$ as $\delta \to 0$ since $\ln(1/|x|)$ belongs to *BMO*. Impossible. □

2.3 On a conjecture of S. Müller

S. Müller has constructed in [32] interesting examples of maps $g \in W^{1,p}(\Omega, \mathbb{R}^N)$ for all $p < N$ such that $\mathrm{Det}(\nabla g)$ is a measure. For such g's he made the following conjecture:

Let M be an $(N-1)$-dimensional manifold, then

$$\mathcal{H}^{N-1}(M \cap S) = 0,$$

where $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure and S is the support of the singular part of the measure $\mathrm{Det}(\nabla g)$.

We present a counterexample to this conjecture. Let $\Omega = (-1, 1)^N$ and let

$$M = \{(x', 0);\ x' \in (-1, 1)^{N-1}\}.$$

Using the dipole technique discussed in Sect. 2.2, we will construct a map g such that

$$\begin{cases} g \in W^{1,p}(\Omega, \mathbb{R}^N) \ \forall\, p < N, \ \ g \in L^\infty(\Omega, \mathbb{R}^N), \\ \operatorname{Det}(\nabla g) \text{ is a measure,} \\ \det(\nabla g) = \text{ the regular part of } \operatorname{Det}(\nabla g) = 0, \\ M \subset S = \text{support of } \operatorname{Det}(\nabla g). \end{cases}$$

We point out however that g is *not* continuous while all the maps discussed in [32] are continuous. S. Müller's conjecture might still be true if one assumes in addition that g is continuous.

Construction of g: Let (P_i) be a dense sequence in M. Consider a dipole f_1 associated to the pair $[P_1, D_1]$ with $\overrightarrow{P_1 D_1} \perp M$ and $\rho_1 = L_1$; we have

$$\operatorname{Det}(\nabla f_1) = \delta_{P_1} - \delta_{D_1}.$$

Next consider a dipole f_2 associated to a pair $[P_2, D_2]$ with $\overrightarrow{P_2 D_2} \perp M$ and $\rho_2 = L_2$ sufficiently small in order to satisfy

$$\operatorname{supp} f_1 \cap \operatorname{supp} f_2 = \emptyset.$$

By induction we construct a sequence (f_n) where f_n is associated to the pair $[P_n, D_n]$ with $\overrightarrow{P_n D_n} \perp M$ and $\rho_n = L_n$ sufficiently small in order to satisfy

$$\operatorname{supp} f_n \cap \operatorname{supp} f_i = \emptyset, \quad \forall 1 \le i \le n-1.$$

From (2.7), we have for all $p < N$,

$$\|\nabla f_n\|_{L^p} \le C(p, N), \quad \forall n.$$

The map

$$g = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n$$

satisfies all the required properties. Indeed note that

$$\operatorname{Det}(\nabla g) = \sum_{n=1}^{\infty} \frac{1}{2^n} (\delta_{P_n} - \delta_{D_n})$$

since the maps f_n have disjoint supports. $\square$

Remark 9 It would be very interesting to decide whether one can construct an example g such that the singular part of $\operatorname{Det}(\nabla g)$ restricted to the manifold M

is truly $(N-1)$-dimensional, say nontrivial and absolutely continuous with respect to the $(N-1)$-dimensional Hausdorff measure.

2.4 Proof of Theorem 2

In the proof of Theorem 2, we will use the following result:

Lemma 1 *Let $N \geq 3$. Then for all $g \in C^1(\bar{\Omega}, \mathbb{R}^N)$ and $\psi \in C^1_c(\Omega)$,*

$$\left| \int_\Omega \det(\nabla g)\psi \right| \leq C_{N,\Omega} \|g_N\|_{BMO} \|\nabla \psi\|_{L^\infty} \prod_{i=1}^{N-1} \|\nabla g_i\|_{L^{N-1}}.$$

Here $g = (g_1, \dots, g_N)$.

Proof Let $G = (G_1, \dots, G_N) = (G', G_N) \in C^1(\mathbb{R}^N, \mathbb{R}^{N-1}) \times C^1(\mathbb{R}^N, \mathbb{R})$ be an extension of g to $\mathbb{R}^N$ such that

$$\|G_i\|_{W^{1,N-1}(\mathbb{R}^N)} \lesssim \|g_i\|_{W^{1,N-1}(\Omega)}, \ \forall 1 \leq i \leq N-1 \quad \text{and}$$
$$\|G_N\|_{BMO(\mathbb{R}^N)} \lesssim \|g_N\|_{BMO(\Omega)}.$$

Then

$$\int_\Omega \det(\nabla g)\psi = \int_{\mathbb{R}^N} \det(\nabla G)\psi.$$

Using (1.3), we have

$$\int_\Omega \det(\nabla g)\psi = -\int_{\mathbb{R}^N} G_N \det(\nabla G', \nabla \psi).$$

Write

$$\det(\nabla G', \nabla \psi) = B \cdot E,$$

where

$$B = \nabla G_1 \quad \text{and} \quad E = (C_{1,1}, \dots, C_{1,N}).$$

Here $C = (C_{i,j})$ is the matrix of cofactors of the matrix $(\nabla G', \nabla \psi)$. It is clear that

$$\operatorname{div} E = 0, \qquad \operatorname{curl} B = 0,$$

and

$$\|E\|_{L^{\frac{N-1}{N-2}}} \lesssim \|\nabla \psi\|_{L^\infty} \prod_{i=1}^{N-1} \|\nabla G_i\|_{L^{N-1}}, \quad \text{and} \quad \|B\|_{L^{N-1}} \lesssim \|\nabla G_1\|_{L^{N-1}}.$$

Applying the result of R. Coifman, P.L. Lions, Y. Meyer, and S. Semmes ([15, Theorem II.1]), we see that $B \cdot E$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^N)$ and the conclusion follows. □

Using Lemma 1, one can prove:

Lemma 2 *Let $N \geq 3$. Then*

(i) *For all $f, g \in C^1(\bar{\Omega}, \mathbb{R}^N)$, and for all $\psi \in C_c^1(\Omega, \mathbb{R})$,*

$$\begin{aligned}&\left| \int_\Omega \det(\nabla f)\psi - \int_\Omega \det(\nabla g)\psi \right| \\ &\quad \leq C_{N,\Omega} \|f - g\|_{BMO} (\|\nabla f\|_{L^{N-1}} + \|\nabla g\|_{L^{N-1}})^{N-1} \|\nabla \psi\|_{L^\infty}.\end{aligned}$$

(ii) *For all $f, g \in C^1(\bar{\Omega}, \mathbb{R}^N)$, and for all $\psi \in C_c^1(\Omega, \mathbb{R})$,*

$$\begin{aligned}&\left| \int_\Omega \det(\nabla f)\psi - \int_\Omega \det(\nabla g)\psi \right| \\ &\quad \leq C_{N,\Omega} \|\nabla f - \nabla g\|_{L^{N-1}} (\|\nabla f\|_{L^{N-1}} \\ &\qquad + \|\nabla g\|_{L^{N-1}})^{N-2} (\|f\|_{BMO} + \|g\|_{BMO}) \|\nabla \psi\|_{L^\infty}.\end{aligned}$$

Proof Set $f = (f_1, \dots, f_N)$ and $g = (g_1, \dots, g_N)$. As in the proof of Theorem 1, write

$$\det(\nabla f) - \det(\nabla g) = \sum_{i=1}^N X_i.$$

Applying (1.3) and Lemma 1 yields

$$\left| \int_\Omega X_i \psi \right| \lesssim \|f_i - g_i\|_{BMO} \|\nabla f\|_{L^{N-1}}^{N-i} \|\nabla g\|_{L^{N-1}}^{i-1} \|\nabla \psi\|_{L^\infty}.$$

This implies (i). To prove (ii), it suffices to note that, by (1.3) and Lemma 1,

$$\begin{aligned}\left| \int_\Omega X_i \psi \right| &\lesssim \|\nabla f_i - \nabla g_i\|_{L^{N-1}} \Big(\|\nabla f\|_{L^{N-1}} + \|\nabla g\|_{L^{N-1}} \Big)^{N-2} \\ &\quad \times \big(\|f\|_{BMO} + \|g\|_{BMO} \big) \|\nabla \psi\|_{L^\infty}.\end{aligned}$$

□

Definition 1 Let $N \geq 3$ and $g \in W^{1,N-1}(\Omega, \mathbb{R}^N) \cap BMO(\Omega, \mathbb{R}^N)$. For each $\psi \in C_c^1(\Omega, \mathbb{R})$, define $\langle \mathrm{Det}(\nabla g), \psi \rangle$ as the limit of $\langle \mathrm{Det}(\nabla g^{(k)}), \psi \rangle$ for a sequence $(g^{(k)}) \subset C^1(\bar{\Omega}, \mathbb{R}^N)$ such that $g^{(k)} \to g$ in $W^{1,N-1}(\Omega)$ and $\sup_k \|g^{(k)}\|_{BMO} < +\infty$. This quantity is well-defined by (ii) of Lemma 2 and the fact that for any $g \in W^{1,N-1}(\Omega, \mathbb{R}^N) \cap BMO(\Omega, \mathbb{R}^N)$ there exists a sequence $g^{(k)} \in C^1(\bar{\Omega}, \mathbb{R}^N)$ such that $g^{(k)} \to g$ in $W^{1,N-1}(\Omega)$ and $\sup_k \|g^{(k)}\|_{BMO} < +\infty$.

Proof of Theorem 2 Assertion (ii) follows immediately from Lemma 2, Definition 1, and the fact that for each $g \in W^{1,N-1}(\Omega, \mathbb{R}^N) \cap BMO(\Omega, \mathbb{R}^N)$ there exists a sequence $(g^{(k)}) \subset C^1(\bar{\Omega}, \mathbb{R}^N)$ such that $g^{(k)} \to g$ in $W^{1,N-1}(\Omega)$ and $\sup_k \|g^{(k)}\|_{BMO} \lesssim \|g\|_{BMO}$. To prove (i) we will use (i) of Lemma 2. Let $(g^{(k)}), (h^{(k)}) \subset C^1(\bar{\Omega}, \mathbb{R}^N)$ be such that $g^{(k)} \to g$ in $W^{1,N-1}(\Omega)$ and $\sup_k \|g^{(k)}\|_{BMO} < +\infty$, $h^{(k)} \to f - g$ in $W^{1,N-1}(\Omega)$ and $\sup_k \|h^{(k)}\|_{BMO} \lesssim \|f - g\|_{BMO}$. Define $f^{(k)} = g^{(k)} + h^{(k)}$. Then by Definition 1, we have

$$\langle \mathrm{Det}(\nabla f), \psi\rangle - \langle \mathrm{Det}(\nabla g), \psi\rangle = \lim_{k\to\infty} \big(\langle \mathrm{Det}(\nabla f^{(k)}), \psi\rangle - \langle \mathrm{Det}(\nabla g^{(k)}), \psi\rangle\big).$$

Applying Lemma 2, we have

$$\begin{aligned}&\big|\langle \mathrm{Det}(\nabla f^{(k)}), \psi\rangle - \langle \mathrm{Det}(\nabla g^{(k)}), \psi\rangle\big|\\ &\quad\lesssim \|h^{(k)}\|_{BMO}(\|\nabla f^{(k)}\|_{L^{N-1}} + \|\nabla g^{(k)}\|_{L^{N-1}})^{N-1}\|\nabla\psi\|_{L^\infty}.\end{aligned}$$

As k goes to infinity, we obtain (i). $\square$

Remark 10 Here is an easy variant of Theorem 2 (proved by the same method). Let $N \geq 2$, $N - 1 < p < +\infty$, and $1 < q < +\infty$ be such that $\frac{N-1}{p} + \frac{1}{q} = 1$. Then, for all $f, g \in W^{1,p}(\Omega, \mathbb{R}^N) \cap BMO(\Omega, \mathbb{R}^N)$ and all $\psi \in C^1_c(\Omega, \mathbb{R})$, we have

(i) $$\begin{aligned}&\big|\langle \mathrm{Det}(\nabla f), \psi\rangle - \langle \mathrm{Det}(\nabla g), \psi\rangle\big|\\ &\quad\leq C_{N,\Omega,p}\|f - g\|_{BMO}(\|\nabla f\|_{L^p} + \|\nabla g\|_{L^p})^{N-1}\|\nabla\psi\|_{L^q}\end{aligned}$$

and

(ii) $$\begin{aligned}&\big|\langle \mathrm{Det}(\nabla f), \psi\rangle - \langle \mathrm{Det}(\nabla g), \psi\rangle\big|\\ &\quad\leq C_{N,\Omega,p}\|\nabla f - \nabla g\|_{L^p}(\|\nabla f\|_{L^p} + \|\nabla g\|_{L^p})^{N-2}\\ &\qquad\times(\|f\|_{BMO} + \|g\|_{BMO})\|\nabla\psi\|_{L^q}.\end{aligned}$$

Remark 11 We do not know whether Theorem 2 holds when $N = 2$. In fact, one may wonder whether there exist a sequence $(g^{(k)}) \subset C^1(\bar{\Omega}, \mathbb{R}^2)$ and a function $\psi \in C^1_c(\Omega, \mathbb{R})$ such that

$$\lim_{k\to\infty} \|g^{(k)}\|_{W^{1,1}} = 0, \qquad \lim_{k\to\infty} \|g^{(k)}\|_{BMO} = 0,$$

and

$$\lim_{k\to\infty}\int_\Omega \det(\nabla g^{(k)})\psi = +\infty.$$

3 Theorems 3 and 4, and optimality results

3.1 Proof of Theorem 3

We begin with the following simple and useful lemma which is inspired from the work of J. Bourgain, H. Brezis, and P. Mironescu in [7, Lemma 3] (see also [22]).

Lemma 3 *Let $N \geq 2$, $g \in C^1(\Omega, \mathbb{R}^N)$, and $\psi \in C^1_c(\Omega, \mathbb{R})$. Then*

$$\int_\Omega \det(\nabla g)\psi = \sum_{i=1}^{N+1}\int_{\Omega\times(0,1)} \mathrm{D}_i(u)\partial_i\varphi\, dx, \tag{3.1}$$

for any extensions $u \in C^1(\Omega \times [0,1), \mathbb{R}^N) \cap C^2(\Omega \times (0,1), \mathbb{R}^N)$ and $\varphi \in C^1_c(\Omega \times [0,1), \mathbb{R})$ of g and ψ. Here

$$\mathrm{D}_i(u) = (-1)^{N-i}\det(\partial_1 u, \dots, \partial_{i-1}u, \partial_{i+1}u, \dots, \partial_{N+1}u), \quad \forall 1 \leq i \leq N,$$

and

$$\mathrm{D}_{N+1}(u) = -\det(\partial_1 u, \dots, \partial_N u).$$

Proof It is important to note that

$$\operatorname{div}\mathrm{D} = 0 \quad \text{in } \Omega \times (0,1).$$

This implies

$$\sum_{i=1}^{N+1}\int_{\Omega\times(0,1)} \mathrm{D}_i\partial_i\varphi = \int_{\partial\left(\Omega\times(0,1)\right)} \varphi(\mathrm{D}\cdot n).$$

Since $\varphi = 0$ for $x \in \partial(\Omega \times (0,1)) \setminus (\Omega \times \{0\})$, the conclusion follows. □

Using Lemma 3, we can prove an important estimate:

Lemma 4 *Let $N \geq 2$ and $g \in C^1(\bar{\Omega}, \mathbb{R}^N)$. Then, for every $\psi \in C^1_c(\Omega)$,*

$$\left|\int_\Omega \det(\nabla g)\psi\right| \leq C_{N,\Omega}\prod_{i=1}^{N}|g_i|_{W^{\frac{N-1}{N},N}}\|\nabla\psi\|_{L^\infty}, \tag{3.2}$$

where $g = (g_1, \ldots, g_N)$*. Moreover, for any* $f, g \in C^1(\bar{\Omega}, \mathbb{R}^N)$ *and* $\psi \in C^1_{\mathrm{c}}(\Omega)$*, we have*

$$\left| \int_\Omega \det(\nabla f)\psi - \int_\Omega \det(\nabla g)\psi \right| \\ \leq C_{N,\Omega} |f - g|_{W^{\frac{N-1}{N},N}} \big(|f|^{N-1}_{W^{\frac{N-1}{N},N}} + |g|^{N-1}_{W^{\frac{N-1}{N},N}} \big) \|\nabla \psi\|_{L^\infty}. \quad (3.3)$$

In this section, $a \lesssim b$ means $a \leq Cb$ for some $C > 0$ depending only on N and Ω, $a \gtrsim b$ means $b \lesssim a$ and $a \approx b$ means $a \lesssim b$ and $b \lesssim a$.

Proof of Lemma 4 Let $\tilde{f}$ and $\tilde{g}$ be extensions of f and g to $\mathbb{R}^N$ such that

$$\|\tilde{f}_i\|_{W^{\frac{N-1}{N},N}(\mathbb{R}^N)} \lesssim \|f_i\|_{W^{\frac{N-1}{N},N}(\Omega)}, \qquad \|\tilde{g}_i\|_{W^{\frac{N-1}{N},N}(\mathbb{R}^N)} \lesssim \|g_i\|_{W^{\frac{N-1}{N},N}(\Omega)},$$

and

$$\|\tilde{f}_i - \tilde{g}_i\|_{W^{\frac{N-1}{N},N}(\mathbb{R}^N)} \lesssim \|f_i - g_i\|_{W^{\frac{N-1}{N},N}(\Omega)},$$

for $i = 1, \ldots, N$ with $f = (f_1, \ldots, f_N)$ and $g = (g_1, \ldots, g_N)$. Let u and v be the extensions by average of $\tilde{g}$ and $\tilde{f}$ to $\Omega \times [0, 1)$ i.e.,

$$u(x, r) = \fint_{B(x,r)} \tilde{g}(y)\, dy \quad \text{and} \quad v(x, r) = \fint_{B(x,r)} \tilde{f}(y)\, dy,$$

where $B(x, r)$ denotes the ball $B(x, r) = \{y \in \mathbb{R}^N; |y - x| < r\}$. We have, by standard trace theory (see e.g. [19]),

$$\|\nabla u_i\|_{L^N(\Omega \times (0,1))} \lesssim \|g_i\|_{W^{\frac{N-1}{N},N}(\Omega)},$$
$$\|\nabla v_i\|_{L^N(\Omega \times (0,1))} \lesssim \|f_i\|_{W^{\frac{N-1}{N},N}(\Omega)},$$

and

$$\|\nabla u_i - \nabla v_i\|_{L^N(\Omega \times (0,1))} \lesssim \|g_i - f_i\|_{W^{\frac{N-1}{N},N}(\Omega)}.$$

Let $\varphi \in C^1_{\mathrm{c}}(\Omega \times [0, 1))$ be an extension of ψ such that $\|\nabla \varphi\|_{L^\infty(\Omega \times [0,1))} \lesssim \|\nabla \psi\|_{L^\infty(\Omega)}$. Since

$$|\mathrm{D}_i(u)| \lesssim \prod_{j=1}^N |\nabla u_j|,$$

and

$$|\mathrm{D}_i(u) - \mathrm{D}_i(v)| \lesssim |\nabla u - \nabla v| \big(|\nabla u|^{N-1} + |\nabla v|^{N-1} \big),$$

it follows from Lemma 3 and Hölder's inequality, that

$$\left|\int_\Omega \det(\nabla g)\psi\right| \le C_{N,\Omega}\prod_{j=1}^N \|g_j\|_{W^{\frac{N-1}{N},N}}\|\nabla\psi\|_{L^\infty},$$

and

$$\left|\int_\Omega \det(\nabla f)\psi - \int_\Omega \det(\nabla g)\psi\right|$$
$$\le C_{N,\Omega}\|f-g\|_{W^{\frac{N-1}{N},N}}\big(\|f\|^{N-1}_{W^{\frac{N-1}{N},N}} + \|g\|^{N-1}_{W^{\frac{N-1}{N},N}}\big)\|\nabla\psi\|_{L^\infty}.$$

Finally, we have

$$\int_\Omega \det(\nabla f)\psi = \int_\Omega \det\Big(\nabla\Big(f - \fint_\Omega f\Big)\Big)\psi \quad \text{and}$$
$$\int_\Omega \det(\nabla g)\psi = \int_\Omega \det\Big(\nabla\Big(g - \fint_\Omega g\Big)\Big)\psi,$$

and therefore we can use the semi-norm $|\ |_{W^{\frac{N-1}{N},N}}$ instead of the norm $\|\ \|_{W^{\frac{N-1}{N},N}}$ in (3.2) and (3.3). □

Based on Lemma 3, we can give a "robust" definition of $\mathrm{Det}(\nabla g)$ when $g \in W^{\frac{N-1}{N},N}(\Omega)$.

Definition 2 Let $N \ge 2$ and $g \in W^{\frac{N-1}{N},N}(\Omega,\mathbb{R}^N)$. For any $\psi \in C^1_c(\Omega,\mathbb{R})$ we define $\langle \mathrm{Det}\,\nabla g, \psi\rangle$ as the limit of $\int_\Omega \det(\nabla g^{(k)})\psi$ for any sequence $(g^{(k)}) \subset C^1(\bar\Omega,\mathbb{R}^N)$ such that $g^{(k)} \to g$ in $W^{\frac{N-1}{N},N}(\Omega)$.

This object is well-defined according to Lemma 4 and the fact that for any $g \in W^{\frac{N-1}{N},N}(\Omega)$, there exists $(g^{(k)}) \subset C^1(\bar\Omega)$ such that $g^{(k)} \to g$ in $W^{\frac{N-1}{N},N}(\Omega)$.

It is clear that Theorem 3 is a consequence of Lemma 3 and Definition 2.

Our next result provides a fundamental representation of the distribution $\mathrm{Det}(\nabla g)$ (which might also serve as an alternative definition for $\mathrm{Det}(\nabla g)$).

Proposition 3 *Let $N \ge 2$, $g \in W^{\frac{N-1}{N},N}(\Omega,\mathbb{R}^N)$, and $\psi \in C^1_c(\Omega,\mathbb{R})$. Then*

$$\langle \mathrm{Det}(\nabla g),\psi\rangle = \sum_{i=1}^{N+1}\int_{\Omega\times(0,1)} \mathrm{D}_i(u)\partial_i\psi, \tag{3.4}$$

for any extensions $u \in W^{1,N}(\Omega \times (0,1), \mathbb{R}^N)$ *and* $\varphi \in C_c^1(\Omega \times [0,1), \mathbb{R})$ *of* g *and* ψ, *where* $\mathrm{D}_i(u) \in L^1(\Omega \times (0,1))$, $1 \le i \le N+1$, *is defined in Lemma* 3.

Proof Let $u^{(k)}$ be a sequence in $C^1(\bar{\Omega} \times [0,1], \mathbb{R}^N)$ such that $u^{(k)} \to u$ in $W^{1,N}(\Omega \times (0,1))$. By trace theory we know that

$$g^{(k)} = u^{(k)}|_{\Omega \times \{0\}} \to g \quad \text{in } W^{\frac{N-1}{N},N}(\Omega, \mathbb{R}^N) \text{ as } k \to \infty.$$

From Definition 2 we deduce that

$$\int_\Omega \det(\nabla g^{(k)})\psi \to \langle \mathrm{Det}(\nabla g), \psi \rangle \quad \text{as } k \to \infty.$$

On the other hand, we have by Lemma 3

$$\int_\Omega \det(\nabla g^{(k)})\psi = \sum_{i=1}^{N+1} \int_{\Omega \times (0,1)} \mathrm{D}_i(u^{(k)})\partial_i \varphi.$$

Passing to the limit as $k \to \infty$ we obtain the desired conclusion. □

From Proposition 3 we can deduce some information about the structure of the distribution $\mathrm{Det}(\nabla g)$ (compare with (1.3) in the case $g \in W^{1,p}(\Omega, \mathbb{R}^N) \cap L^q(\Omega, \mathbb{R}^N)$ with $\frac{N-1}{p} + \frac{1}{q} = 1$).

Corollary 2 *Let* $N \ge 2$ *and* $g \in W^{\frac{N-1}{N},N}(\Omega, \mathbb{R}^N)$. *Then there exist* $h = (h_i) \in [L^1(\Omega)]^N$ *such that* $\|h\|_{L^1} \le C_{\Omega,N} \|g\|^N_{W^{\frac{N-1}{N},N}}$ *and*

$$\mathrm{Det}(\nabla g) = \sum_{i=1}^N \frac{\partial}{\partial x_i} h_i \quad \text{in } \mathcal{D}'(\Omega).$$

Moreover, for $g^{(1)}$ *and* $g^{(2)} \in W^{\frac{N-1}{N},N}(\Omega)$, *there exist* $h^{(1)}$ *and* $h^{(2)} \in [L^1(\Omega)]^N$ *such that* $\|h^{(1)} - h^{(2)}\|_{L^1} \le C_{\Omega,N} \|g^{(1)} - g^{(2)}\|_{W^{\frac{N-1}{N},N}} \times (\|g^{(1)}\|_{W^{\frac{N-1}{N},N}} + \|g^{(2)}\|_{W^{\frac{N-1}{N},N}})^{N-1}$ *and*

$$\mathrm{Det}(\nabla g^{(m)}) = \sum_{i=1}^N \frac{\partial}{\partial x_i} h_i^{(m)} \quad \text{in } \mathcal{D}'(\Omega) \text{ for } m = 1, 2.$$

Proof We only prove the first statement of Corollary 2. The second statement follows by the same method. Let u be the extension of g as in Lemma 4. Then

$$\|u\|_{W^{1,N}(\Omega \times (0,1))} \le C_{\Omega,N} \|g\|_{W^{\frac{N-1}{N},N}(\Omega)}. \tag{3.5}$$

Let $\psi \in C_c^1(\Omega, \mathbb{R})$ and $\zeta \in C^1([0,1], \mathbb{R})$ be such that $\zeta = 1$ on $[0, 1/4]$ and $\zeta = 0$ on $[1/2, 1]$. Set $\varphi(x, x_{N+1}) = \psi(x)\zeta(x_{N+1})$. By Proposition 3, we have

$$\langle \mathrm{Det}(\nabla g), \psi \rangle = \sum_{i=1}^{N} \int_{\Omega \times (0,1)} \mathrm{D}_i(u) \zeta \frac{\partial \psi}{\partial x_i} + \int_{\Omega \times (0,1)} \mathrm{D}_{N+1}(u) \psi \frac{\partial \zeta}{\partial x_{N+1}}.$$

Set

$$\tilde{h}_i(x) = \int_0^1 \mathrm{D}_i(u) \zeta(x_{N+1}) \, dx_{N+1}, \qquad \forall 1 \le i \le N,$$

and

$$\tilde{h}_{N+1}(x) = \int_0^1 \mathrm{D}_{N+1}(u) \zeta'(x_{N+1}) \, dx_{N+1}.$$

By Fubini we know that $\tilde{h}_i$ belongs to $L^1(\Omega)$ for $i = 1, \ldots, N+1$. Moreover we have

$$\langle \mathrm{Det}(\nabla g), \psi \rangle = \sum_{i=1}^{N} \int_{\Omega} \tilde{h}_i \frac{\partial \psi}{\partial x_i} + \int_{\Omega} \tilde{h}_{N+1} \psi,$$

i.e.,

$$\mathrm{Det}(\nabla g) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \tilde{h}_i + \tilde{h}_{N+1}.$$

The conclusion follows from (3.5) by writing $\tilde{h}_{N+1}$ as a divergence of an L^1 vector-field. $\square$

Remark 12 In view of Corollary 2, the conclusion of Proposition 2 remains valid for every $g \in W^{\frac{N-1}{N}, N}(\Omega)$, in particular one cannot have $\mathrm{Det}(\nabla g) = \frac{\partial}{\partial x_N} \delta_a$ for some $a \in \Omega$ if $g \in W^{\frac{N-1}{N}, N}(\Omega)$.

3.2 Optimality results

In this section, $a \lesssim b$ means $a \le Cb$ for some $C > 0$ independent of k, $a \gtrsim b$ means $b \lesssim a$ and $a \approx b$ means $a \lesssim b$ and $b \lesssim a$.

3.2.1 Proof of Theorem 4

Theorem 4 is consequence of the following lemma as explained in the Introduction.

Lemma 5 *Let $N \geq 2$, $s \in (0,1)$, and $p \in (1,\infty)$ be such that*

(i) *either*

$$s + \frac{1}{N} < \max\left\{1, \frac{N}{p}\right\},$$

(ii) *or*

$$s = 1 - \frac{1}{N} \quad \textit{and} \quad p > N.$$

Then there exist a sequence $(g^{(k)}) \subset C^1(\bar{\Omega}, \mathbb{R}^N)$ and a function $\psi \in C^1_{\mathrm{c}}(\Omega, \mathbb{R})$ such that $\lim_{k\to\infty} \|g^{(k)}\|_{W^{s,p}} = 0$ and

$$\lim_{k\to\infty} \int_\Omega \det(\nabla g^{(k)})\psi = +\infty.$$

Remark 13 Let $s = \frac{N-1}{N}$ and $p \in (1, +\infty)$. We deduce from Lemma 5 and Theorem 3 that $W^{s,p}(\Omega) \subset W^{s,N}(\Omega)$ if and only if $p = N$. Indeed, suppose that $W^{s,p}(\Omega) \subset W^{\frac{N-1}{N},N}(\Omega)$. Consider the case $p < N$. Applying (i) of Lemma 5 and the closed graph theorem we have

$$\|g^{(k)}\|_{W^{s,N}} \leq C\|g^{(k)}\|_{W^{s,p}} \to 0 \quad \text{as } k \to \infty,$$

and

$$\lim_{k\to\infty} \int_\Omega \det(\nabla g^{(k)})\psi = +\infty.$$

On the other hand, we deduce from Theorem 3 that

$$\left|\int_\Omega \det(\nabla g^{(k)})\psi\right| \leq C\|g^{(k)}\|^N_{W^{s,N}}\|\nabla\psi\|_{L^\infty} \to 0 \quad \text{as } k \to \infty,$$

which contradicts the previous assertion.

When $p > N$, we apply (ii) of Lemma 5 and obtain again a contradiction. P. Mironescu [28] has established the same property in the general case: let $s \in (0,1)$ and $p, q \in (1, +\infty)$, then $W^{s,p}(\Omega) \subset W^{s,q}(\Omega)$ if and only if $p = q$.

There is a *sharp difference* with the situation in the Bessel potential spaces H^s_p. For such spaces it is known (see e.g. [41, Theorem on page 196]) that $H^s_p(\Omega) \subset H^s_q(\Omega)$ when $p \geq q$. As a consequence $\mathrm{Det}(\nabla g)$ is well-defined when $g \in H_p^{\frac{N-1}{N}}(\Omega)$ and $p > N$, since $H_p^{\frac{N-1}{N}}(\Omega) \subset H_N^{\frac{N-1}{N}}(\Omega) \subset W^{\frac{N-1}{N},N}(\Omega)$; by contrast $\mathrm{Det}(\nabla g)$ is meaningless on the space $W^{\frac{N-1}{N},p}(\Omega)$ when $p > N$.

Proof of Lemma 5 Without loss of generality, one may assume that $(-4,4)^N \subset \Omega$. We distinguish three cases:

Case 1: $p \le N$ and $s + 1/N < N/p$.
Case 2: $p > N$ and $s + 1/N < 1$.
Case 3: $p > N$ and $s = 1 - 1/N$.

Case 1: We use the same notation as in the proof of Remark 1, then we set

$$h_\varepsilon(x) = \varepsilon^{-\frac{1}{N}} g(x/\varepsilon).$$

We know (see (2.20)) that

$$\int_\Omega \det(\nabla h_\varepsilon)\psi = \sum_{i=1}^N \alpha_i \frac{\partial \psi}{\partial x_i}(0) + O(\varepsilon) \quad \text{with } \alpha_1 = 1.$$

On the other hand,

$$\begin{aligned} |h_\varepsilon|^p_{W^{s,p}} &= \frac{1}{\varepsilon^{\frac{p}{N}}} \int_\Omega \int_\Omega \frac{|g(x/\varepsilon) - g(y/\varepsilon)|^p}{|x-y|^{N+sp}}\, dx\, dy \\ &\approx \varepsilon^{N-sp-p/N} \to 0 \quad \text{as } \varepsilon \to 0. \end{aligned}$$

Indeed, recall that $s + 1/N < N/p$. To obtain the desired conclusion take $g^{(k)}(x) = \varepsilon^{-\gamma} h_\varepsilon(x)$ with $0 < \gamma$ sufficiently small and $\varepsilon = 1/k$, and then choose any function $\psi \in C^\infty_{\mathrm{c}}(\Omega)$ such that $\frac{\partial \psi}{\partial x_1}(0) > 0$ and $\frac{\partial \psi}{\partial x_i}(0) = 0$ for $i = 2, \dots, N$.

Case 2 can be deduced from Case 3. However the proof of Case 2 is simple, while the proof of Case 3 is tricky. Since the proof of Case 3 borrows some ideas from the proof of Case 2, we have included both proofs for the convenience of the reader.

Case 2: Let $0 < s < \alpha < \frac{N-1}{N}$. For $k \gg 1$, define $g^{(k)} = (g_1^{(k)}, \dots, g_N^{(k)}) : \Omega \to \mathbb{R}^N$ as follows

$$g_i^{(k)}(x) = k^{-\alpha} \sin(kx_i), \quad \forall 1 \le i \le N-1,$$

and

$$g_N^{(k)}(x) = k^{-\alpha} x_N \prod_{i=1}^{N-1} \cos(kx_i).$$

We have

$$\det(\nabla g^{(k)}) = k^{(N-1)(1-\alpha)-\alpha} \prod_{i=1}^{N-1} \cos^2(kx_i) \ge 0. \tag{3.6}$$

Since $\|\nabla g^{(k)}\|_{L^\infty} \lesssim k^{1-\alpha}$ and $\|g^{(k)}\|_{L^\infty} \lesssim k^{-\alpha}$, it follows by interpolation that

$$|g^{(k)}|_{C^{0,\alpha}(\bar{\Omega})} \lesssim 1. \tag{3.7}$$

Let $\psi \in C^1_{\mathrm{c}}(\Omega, \mathbb{R})$ be such that $\operatorname{supp}\psi \subset (1/5, 4/5)^N$, $\psi \geq 0$, $\psi = 1$ on $(1/4, 3/4)^N$. We have

$$\begin{aligned}\int_\Omega &\det(\nabla g^{(k)})\psi \, dx \\ &\geq \int_{(1/4,3/4)^N} k^{(N-1)(1-\alpha)-\alpha} \prod_{i=1}^{N-1} \cos^2(kx_i)\, dx \gtrsim k^{(N-1)(1-\alpha)-\alpha}. \end{aligned}\tag{3.8}$$

It follows from (3.8) that

$$\lim_{k\to\infty} \int_\Omega \det(\nabla g^{(k)})\, \psi \, dx = +\infty.$$

Since $\alpha > s$, we have $\|g^{(k)}\|_{W^{s,p}} \lesssim \|g^{(k)}\|_{C^{0,\alpha}(\bar{\Omega})}$ and the conclusion follows.

Case 3: Fix $k \gg 1$. Let

$$n_\ell = k^{N^2} 8^\ell \quad \text{for } 1 \leq \ell \leq k. \tag{3.9}$$

It is clear that

$$n_{\ell+1} \geq 2n_\ell, \quad \forall \ell = 1, \ldots, k-1, \tag{3.10}$$

$$\begin{aligned}&\{n_\ell;\ \ell = 1, \ldots, k\} \cap \{z \in \mathbb{R};\ 2^{m-1} \leq |z| < 2^m\} \text{ has at most one element,} \\ &\forall m \in \mathbb{N},\end{aligned}\tag{3.11}$$

and

$$\min_{i \neq j} |n_i - n_j| \geq k^{\frac{N^2}{N-1}}. \tag{3.12}$$

Define $g^{(k)} = (g_1^{(k)}, \ldots, g_N^{(k)}) : \Omega \to \mathbb{R}^N$ as follows

$$g_i^{(k)}(x) = \sum_{\ell=1}^k \frac{1}{n_\ell^{\frac{N-1}{N}} (\ell+1)^{1/N}} \sin(n_\ell x_i), \quad \forall 1 \leq i \leq N-1, \tag{3.13}$$

and

$$g_N^{(k)}(x) = x_N \sum_{\ell=1}^k \frac{1}{n_\ell^{\frac{N-1}{N}} (\ell+1)^{1/N}} \prod_{i=1}^{N-1} \cos(n_\ell x_i). \tag{3.14}$$

Fix $\psi_1 \in C^1_c(\mathbb{R})$ such that $\operatorname{supp}\psi_1 \subset (0,1)$, $\psi_1 \geq 0$, $\psi_1 = 1$ in $(1/4, 3/4)$. Define $\psi : \Omega \to \mathbb{R}$ by $\psi(x_1, \ldots, x_N) = \prod_{i=1}^N \psi_1(x_i)$.

We claim that

$$\int_\Omega \det(\nabla g^{(k)})\psi \gtrsim \ln k \tag{3.15}$$

and

$$\|g^{(k)}\|^p_{W^{\frac{N-1}{N},p}} \approx 1. \tag{3.16}$$

Assuming that (3.15) and (3.16) hold, we deduce that $h^{(k)} = (\ln k)^{-1/(2N)} g^{(k)}$ satisfies all the requirements. Hence it remains to prove (3.15) and (3.16).

Step 1: Proof of (3.15).

From (3.13) and (3.14), it is clear that

$$\det(\nabla g^{(k)}) = \left[\prod_{i=1}^{N-1}\left(\sum_{\ell_i=1}^{k} \frac{n_{\ell_i}^{\frac{1}{N}}}{(\ell_i+1)^{1/N}} \cos(n_{\ell_i} x_i)\right)\right] \times \left(\sum_{\ell_N=1}^{k} \frac{1}{n_{\ell_N}^{\frac{N-1}{N}}(\ell_N+1)^{1/N}} \prod_{j=1}^{N-1} \cos(n_{\ell_N} x_j)\right).$$

This implies

$$\det(\nabla g^{(k)}) = \sum_{\ell=1}^{k} \frac{1}{(\ell+1)} \prod_{i=1}^{N-1} \cos^2(n_\ell x_i) + \sum_{(\ell_1,\ldots,\ell_N)\neq(\ell,\ldots,\ell) \text{ for } \ell=1,\ldots,k} \frac{1}{n_{\ell_N}^{\frac{N-1}{N}}(\ell_N+1)^{1/N}} \times \prod_{i=1}^{N-1}\left[\frac{n_{\ell_i}^{\frac{1}{N}}}{(\ell_i+1)^{1/N}} \cos(n_{\ell_i} x_i)\cos(n_{\ell_N} x_i)\right]. \tag{3.17}$$

On the other hand,

$$\left|\frac{1}{n_{\ell_N}^{\frac{N-1}{N}}(\ell_N+1)^{1/N}}\int_\Omega \psi(x)\prod_{i=1}^{N-1}\Big[\frac{n_{\ell_i}^{\frac{1}{N}}}{(\ell_i+1)^{1/N}}\cos(n_{\ell_i}x_i)\cos(n_{\ell_N}x_i)\Big]\,dx\right|$$

$$\lesssim \frac{1}{n_{\ell_N}^{\frac{N-1}{N}}(\ell_N+1)^{1/N}}\prod_{i=1}^{N-1}\frac{n_{\ell_i}^{\frac{1}{N}}}{(\ell_i+1)^{1/N}}$$

$$\times\left|\int_0^1 \psi_1(x_i)\cos(n_{\ell_i}x_i)\cos(n_{\ell_N}x_i)\,dx_i\right| \tag{3.18}$$

and

$$\left|\int_0^1 \psi_1(x_i)\cos(n_{\ell_i}x_i)\cos(n_{\ell_N}x_i)\,dx_i\right| \lesssim \min\{1/|n_{\ell_i}-n_{\ell_N}|,1\}. \tag{3.19}$$

Since $\ell_j\ge 1$ for $1\le j\le N$, it follows from (3.18) and (3.19) that

$$\left|\frac{1}{n_{\ell_N}^{\frac{N-1}{N}}(\ell_N+1)^{1/N}}\int_\Omega \psi(x)\prod_{i=1}^{N-1}\frac{n_{\ell_i}^{\frac{1}{N}}}{(\ell_i+1)^{1/N}}\cos(n_{\ell_i}x_i)\cos(n_{\ell_N}x_i)\,dx\right|$$

$$\lesssim \prod_{i=1}^{N-1}\Bigg[\frac{n_{\ell_i}^{\frac{1}{N}}}{n_{\ell_N}^{\frac{1}{N}}}\min\{1/|n_{\ell_i}-n_{\ell_N}|,1\}\Bigg]. \tag{3.20}$$

If $(\ell_1,\ldots,\ell_N)\neq(\ell,\ldots,\ell)$ for all $\ell=1,\ldots,k$, then there exists $1\le i\le N-1$ such that $\ell_i\neq\ell_N$. Hence from (3.20) and (3.10), we have, if $(\ell_1,\ldots,\ell_N)\neq(\ell,\ldots,\ell)$ for all $\ell=1,\ldots,k$,

$$\left|\frac{1}{n_{\ell_N}^{\frac{N-1}{N}}(\ell_N+1)^{1/N}}\int_\Omega \psi(x)\prod_{i=1}^{N-1}\frac{n_{\ell_i}^{\frac{1}{N}}}{(\ell_i+1)^{1/N}}\cos(n_{\ell_i}x_i)\cos(n_{\ell_N}x_i)\,dx\right|$$

$$\lesssim \frac{n_{\ell_i}^{\frac{1}{N}}}{n_{\ell_N}^{\frac{1}{N}}}\frac{1}{|n_{\ell_i}-n_{\ell_N}|}\lesssim\frac{1}{|n_{\ell_i}-n_{\ell_N}|^{\frac{N-1}{N}}}, \tag{3.21}$$

for some $1\le i\le N-1$.

Since

$$\int_\Omega \psi\prod_{i=1}^{N-1}\cos^2(n_\ell x_i)\gtrsim 1,\quad \forall\ell=1,\ldots,k,$$

we deduce from (3.17) and (3.21) that

$$\int_\Omega \det(\nabla g^{(k)})\psi \gtrsim \sum_{\ell=1}^{k} \frac{1}{(\ell+1)} - Ck^N \max_{i,j} \frac{1}{|n_i - n_j|^{\frac{N-1}{N}}}.$$

Claim (3.15) now follows from (3.12).

Step 2: Proof of (3.16).

Let $\mathbb{T}^N = [-\pi, \pi]^N$ be the N-dimensional torus. We will prove that

$$\|g^{(k)}\|_{W^{\frac{N-1}{N},p}(\mathbb{T}^N)} \approx 1. \tag{3.22}$$

This will imply (3.16) since

$$\|g^{(k)}\|_{W^{\frac{N-1}{N},p}(\Omega)} \approx \|g^{(k)}\|_{W^{\frac{N-1}{N},p}(\mathbb{T}^N)}.$$

For this purpose we define

$$R_0 = \{0 = (0, \dots, 0) \in \mathbb{Z}^N\},$$
$$R_j = \left\{ r = (r_1, \dots, r_N) \in \mathbb{Z}^N;\ 2^{j-1} \le \max_{m=1,\dots,N} |r_m| < 2^j \right\}, \quad \forall j \in \mathbb{N}_+.$$

We recall (see e.g. [38, Theorems on pages 167 and 168]) that for $0 < s < 1$ and $1 < p < +\infty$,

$$\begin{aligned} C_1 \sum_{j\in\mathbb{N}} 2^{spj} \Big\| \sum_{r\in R_j} a_r e^{ir\cdot x} \Big\|_{L^p(\mathbb{T}^N)}^p &\le \Big\| \sum_{r\in\mathbb{Z}^N} a_r e^{ir\cdot x} \Big\|_{W^{s,p}(\mathbb{T}^N)}^p \\ &\le C_2 \sum_{j\in\mathbb{N}} 2^{spj} \Big\| \sum_{r\in R_j} a_r e^{ir\cdot x} \Big\|_{L^p(\mathbb{T}^N)}^p, \end{aligned} \tag{3.23}$$

for some positive constants C_1 and C_2 depending only on N, s, and p.

We claim that

$$\|g_i^{(k)}\|_{W^{\frac{N-1}{N},p}(\mathbb{T}^N)}^p \approx \sum_{\ell=1}^{k} \frac{1}{(1+\ell)^{p/N}} \|\sin(n_\ell x_i)\|_{L^p(\mathbb{T}^N)}^p, \quad \forall 1 \le i \le N-1. \tag{3.24}$$

Indeed, since $\sin(n_\ell x_i) = \frac{1}{2i}[e^{in_\ell x_i} - e^{-in_\ell x_i}]$, we have $g_i^{(k)} = \sum_{r\in\mathbb{Z}^N} a_r e^{ir\cdot x}$ for some (a_r). Moreover from (3.11), for each $j \in \mathbb{N}$, either $\sum_{r\in R_j} a_r e^{ir\cdot x} = 0$ or $\sum_{r\in R_j} a_r e^{ir\cdot x} = \frac{1}{n_\ell^{\frac{N-1}{N}} (\ell+1)^{1/N}} \sin(n_\ell x_i)$ for some ℓ with $n_\ell \approx 2^j$. Therefore, (3.24) follows from (3.23).

Using the fact that $p > N$ we deduce from (3.24) that

$$\|g_i^{(k)}\|^p_{W^{\frac{N-1}{N},p}(\mathbb{T}^N)} \approx 1, \quad \forall 1 \le i \le N-1. \tag{3.25}$$

Similarly, we have

$$\|g_N^{(k)}/x_N\|^p_{W^{\frac{N-1}{N},p}(\mathbb{T}^N)} \approx \sum_{\ell=1}^{k} \frac{1}{(1+\ell)^{p/N}} \left\| \prod_{\ell=1}^{N-1} \cos(n_\ell x_i) \right\|^p_{L^p(\mathbb{T}^N)} \approx 1. \tag{3.26}$$

Assertion (3.22) now follows from (3.25) and (3.26). □

Remark 14 Statement (i) of Lemma 5 follows from Cases 1 and 2, statement (ii) follows from Case 3.

3.2.2 *Optimality of Corollary 1*

Corollary 1 is optimal in the following sense:

Proposition 4 *Let $N \ge 2$. There exist a sequence $(g^{(k)}) \subset C^1(\bar{\Omega}, \mathbb{R}^N)$ and a function $\psi \in C_c^1(\Omega, \mathbb{R})$ such that $\lim_{k\to\infty} \|g^{(k)}\|_{C^{0,\frac{N-1}{N}}(\bar{\Omega})} = 0$, $\sup_k \|g^{(k)}\|_{W^{\frac{N-1}{N},N}} < +\infty$, and*

$$\lim_{k\to\infty} \int_\Omega \det(\nabla g^{(k)})\psi > 0.$$

Proof We use the same notation as in the proof of Lemma 5, Case 3. As in the proof of the Case 3 of Lemma 5, we have

$$\|g^{(k)}\|^N_{W^{\frac{N-1}{N},N}} \approx \sum_{\ell=1}^{k} \frac{1}{(1+\ell)} \approx \ln k. \tag{3.27}$$

We claim that

$$\|g^{(k)}\|_{C^{0,\frac{N-1}{N}}(\bar{\Omega})} \lesssim 1. \tag{3.28}$$

Assuming (3.28) holds, we deduce from (3.15), (3.27), and (3.28) that $h^{(k)} = (\ln k)^{-1/N} g^{(k)}$ satisfies all the requirements. Hence it remains to prove (3.28).

Let $\Psi \in C_c^\infty(\mathbb{R}^N)$ be such that Ψ is radial,

$$\operatorname{supp}\Psi \subset \{x \in \mathbb{R}^N;\ 1/2 \le |x| \le 2\} \quad \text{and}$$
$$\Psi > 0 \text{ in } \{x \in \mathbb{R}^N;\ 1/\sqrt{2} \le |x| \le \sqrt{2}\}.$$

Define $\Phi_j(x) = \Psi(2^{-j}x)(\sum_{m=-\infty}^{\infty}\Psi(2^{-m}x))^{-1}$ if $j = 1, 2, \ldots$ and $\Phi_0(x) = 1 - \sum_{j=1}^{\infty}\Phi_j(x)$. We recall (see e.g. [38, Theorem on 168]) that for $0 < s < 1$,

$$\begin{aligned} & C_1 \sup_{j\in\mathbb{N}} 2^{sj}\Big\| \sum_{r\in\mathbb{Z}^N} \Phi_j(r) a_r e^{ir\cdot x}\Big\|_{L^\infty(\mathbb{T}^N)} \\ & \quad \le \Big\| \sum_{r\in\mathbb{Z}^N} a_r e^{ir\cdot x}\Big\|_{C^s(\mathbb{T}^N)} \\ & \quad \le C_2 \sup_{j\in\mathbb{N}} 2^{sj}\Big\| \sum_{r\in\mathbb{Z}^N} \Phi_j(r) a_r e^{ir\cdot x}\Big\|_{L^\infty(\mathbb{T}^N)}, \end{aligned} \tag{3.29}$$

for some positive constants C_1 and C_2 depending only on N and s.

We claim that

$$\begin{aligned} \|g_i^{(k)}\|_{C^{\frac{N-1}{N}}(\mathbb{T}^N)} &\lesssim \sup_{\ell=1,\ldots,k} \frac{1}{(1+\ell)^{1/N}} \|\sin(n_\ell x_i)\|_{L^\infty(\mathbb{T}^N)} \\ &\lesssim 1, \quad \forall 1 \le i \le N-1, \end{aligned} \tag{3.30}$$

Indeed, since $\sin(n_\ell x_i) = \frac{1}{2i}[e^{in_\ell x_i} - e^{-in_\ell x_i}]$, we have $g_i^{(k)} = \sum_{r\in\mathbb{Z}^N} a_r e^{ir\cdot x}$ for some (a_r). Moreover from (3.9), for each $j \in \mathbb{N}$, either $\sum_{r\in\mathbb{Z}^N}\Phi_j(r)a_r e^{ir\cdot x} = 0$ or $\sum_{r\in\mathbb{Z}^N}\Phi_j(r)a_r e^{ir\cdot x} = \frac{\Phi_j((n_\ell,0,\ldots,0))}{n_\ell^{\frac{N-1}{N}}(\ell+1)^{1/N}}\sin(n_\ell x_i)$ for some ℓ with $n_\ell \approx 2^j$ (because Φ_j is radial). Since $|\Phi_j| \le 1$, (3.30) follows from (3.29).

Similarly,

$$\|g_N^{(k)}/x_N\|_{C^{\frac{N-1}{N}}(\mathbb{T}^N)} \lesssim \sup_{\ell=1,\ldots,k} \frac{1}{(1+\ell)^{1/N}} \Big\| \prod_{\ell=1}^{N-1} \cos(n_\ell x_i)\Big\|_{L^\infty(\mathbb{T}^N)} \lesssim 1.$$

Hence by the same method used in the proof of the Case 3 of Lemma 5, we have

$$\|g^{(k)}\|_{C^{0,\frac{N-1}{N}}(\bar{\Omega})} \lesssim 1. \qquad \square$$

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Appendix: An interpolation inequality

Lemma A.1 *Let* $N \geq 1$, $\theta \in (0,1)$, $s > 0$, *and* $p > 1$. *Suppose that* $g \in W^{s,p}(\mathbb{R}^N) \cap BMO(\mathbb{R}^N)$, *then* $g \in W^{\theta s, p/\theta}(\mathbb{R}^N)$ *and*

$$\|g\|_{W^{\theta s,p/\theta}(\mathbb{R}^N)} \leq c(N,p,\theta,s)\|g\|^{\theta}_{W^{s,p}(\mathbb{R}^N)}\|g\|^{1-\theta}_{BMO(\mathbb{R}^N)}. \tag{A.1}$$

Here we use the following *BMO*-norm:

$$\|g\|_{BMO(\mathbb{R}^N)} := \sup_{|Q|<1} \fint_Q \fint_Q |g(x) - g(y)|\,dx\,dy + \sup_{|Q|>1} \fint_Q |g|\,dx,$$

where Q denotes a cube of $\mathbb{R}^n$, $|Q|$ denotes the volume of Q and $\fint_Q := \frac{1}{|Q|}\int_Q$.

The proof of Lemma A.1 uses two basic ingredients:

(i) An estimate due to Oru [33]. The proof of this estimate is not readily available; we refer the reader to the proof reproduced in [11]. Related results also appeared in [14] and [13].
(ii) A characterization of *BMO* functions in terms of their Littlewood-Paley decomposition (see e.g., [41]).

Proof of Lemma A.1 In this proof we will use the standard notion of Littlewood-Paley theory (see e.g. [41]). Let $\Psi \in C_c^\infty(\mathbb{R}^N)$ be such that

$$\operatorname{supp}\Psi \subset \{x \in \mathbb{R}^N;\, 1/2 \leq |x| \leq 2\} \quad \text{and}$$
$$\Psi > 0 \text{ in } \{x \in \mathbb{R}^N;\, 1/\sqrt{2} \leq |x| \leq \sqrt{2}\}.$$

Define $\Phi_j(x) = \Psi(2^{-j}x)(\sum_{k=-\infty}^{\infty}\Psi(2^{-k}x))^{-1}$ if $j = 1, 2, \ldots$ and $\Phi_0(x) = 1 - \sum_{j=1}^{\infty}\Phi_j(x)$. For $u \in \mathcal{S}'(\mathbb{R}^N)$ (the space of tempered distributions), set

$$u_j = \mathcal{F}^{-1}\Phi_j\mathcal{F}u,$$

where $\mathcal{F}$ denotes the Fourier transform. We have (see e.g., [41, page 51])

$$\|g\|_{W^{s,p}} \approx \begin{cases} \|g\|_{\tilde{F}^s_{p,p}} & \text{if } s \notin \mathbb{N}_+, \\ \|g\|_{\tilde{F}^s_{p,2}} & \text{otherwise,} \end{cases}$$

and

$$\|g\|_{BMO} \approx \|g\|_{\tilde{F}^0_{\infty,2}}.$$

Here

$$\|g\|_{\tilde{F}^s_{p,q}} := \left(\int_{\mathbb{R}^N} \left(\sum_{j=0}^{\infty} 2^{sqj} |g_j(x)|^q \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}},$$

where $g_j := \mathcal{F}^{-1}\Phi_j \mathcal{F} g$, for $-\infty < s < +\infty$, $0 < p \le +\infty$, and $0 < q \le +\infty$ with the usual notation for $p = +\infty$ or $q = +\infty$. On the other hand, [11, Lemma 3.1] asserts that

$$\|f\|_{\tilde{F}^{s_0}_{p_0,q_0}} \lesssim \|f\|^{\theta}_{\tilde{F}^{s_1}_{p_1,q_1}} \|f\|^{1-\theta}_{\tilde{F}^{s_2}_{p_2,q_2}}, \tag{A.2}$$

for $-\infty < s_1, s_2 < +\infty$, $0 < q_0, q_1, q_2 \le +\infty$, $0 < p_1, p_2 \le +\infty$, $0 < \theta < 1$, and (s_0, p_0) such that

$$s_0 = \theta s_1 + (1-\theta)s_2,$$
$$\frac{1}{p_0} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}.$$

Applying (A.2) with $(s_1, p_1, q_1) = (s, p, p)$ if $s \notin \mathbb{N}_+$ and $(s_1, p_1, q_1) = (s, p, 2)$ otherwise, $(s_2, p_2, q_2) = (0, +\infty, +\infty)$, and $(s_0, p_0, q_0) = (\theta s, p/\theta, p/\theta)$ if $\theta s \notin \mathbb{N}_+$ and $(s_0, p_0, q_0) = (\theta s, p/\theta, 2)$ otherwise, we have

$$\|g\|_{W^{\theta s, p/\theta}} \le c(N, p, \theta, s) \|g\|^{\theta}_{W^{s,p}} \|g\|^{1-\theta}_{\tilde{F}^0_{+\infty,+\infty}}.$$

Since

$$\|g\|_{\tilde{F}^0_{+\infty,+\infty}} = \sup_x \sup_j |g_j(x)| \le \sup_x \left(\sum_{j=0}^{\infty} |g_j(x)|^2 \right)^{\frac{1}{2}} = \|g\|_{\tilde{F}^0_{\infty,2}} \approx \|g\|_{BMO},$$

the conclusion follows. □

Lemma A.2 *Let $N \ge 1$, $\theta \in (0,1)$, $s > 0$, $p > 1$, and Ω be a smooth bounded open subset of $\mathbb{R}^N$. Suppose that $g \in W^{s,p}(\Omega) \cap BMO(\Omega)$ then $g \in W^{\theta s, p/\theta}(\Omega)$ and*

$$\|g\|_{W^{\theta s, p/\theta}} \le c(N, p, \theta, s, \Omega) \|g\|^{\theta}_{W^{s,p}} \|g\|^{1-\theta}_{BMO}.$$

Proof Let G be an extension of g to $\mathbb{R}^N$ such that $\|G\|_{W^{s,p}(\mathbb{R}^N)} \lesssim \|g\|_{W^{s,p}(\Omega)}$ and $\|G\|_{BMO(\mathbb{R}^N)} \lesssim \|g\|_{BMO(\Omega)}$. By Lemma A.1, $G \in W^{\theta s, p/\theta}(\mathbb{R}^N)$ and

$$\|G\|_{W^{\theta s, p/\theta}(\mathbb{R}^N)} \le c(N, p, \theta, s, \Omega) \|G\|^{\theta}_{W^{s,p}(\mathbb{R}^N)} \|G\|^{1-\theta}_{BMO(\mathbb{R}^N)}.$$

The conclusion follows. □

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